Contents

Chapter 1

$ax^2 + bx + c$

A quadratic polynomial is a polynomial of the form of $ax^2 + bx + c$ for some real numbers a, b, c with $a \neq 0$. Furthermore, a is called the *leading coefficient* and c is called the *constant term.*

Consider the polynomial $P(x) = ax^2 + bx + c$, where $a \neq 0$. Assume that $a > 0$. Then,

$$P(x) = a\left(x + \frac{b}{2a}\right)^2 - \frac{b^2 - 4ac}{4a}.$$

Let $\Delta = b^2 - 4ac$. Then,

$$P(x) = a\left(x + \frac{b}{2a}\right)^2 - \frac{\Delta}{4a}. \tag{1}$$

(i) If $\Delta < 0$, then $P(x) > 0$ for all real numbers x, so $P(x) = 0$ has no real solutions.

(ii) If $\Delta = 0$, then $P(x) = 0$ reduces to the equation $x + \dfrac{b}{2a} = 0$ and so $P(x) = 0$ has exactly one real solution

$$x = -\frac{b}{2a}.$$

(iii) If $\Delta > 0$, then the equation $P(x) = 0$ becomes the equation $\left(x + \dfrac{b}{2a}\right)^2 = \dfrac{\Delta}{4a^2}$. This equation has two distinct real solutions

$$x_{1,2} = \frac{-b \pm \sqrt{\Delta}}{2a}.$$

We call Δ the *discriminant* of the quadratic polynomial $ax^2 + bx + c$.

One more thing that you also should keep in mind is the graphical representation of quadratic equations.
If $P(x) = ax^2 + bx + c$ is a quadratic, then the graph $y = P(x)$ is a parabola. This parabola opens upward if $a > 0$ and downward if $a < 0$. The parabola is symmetric about the vertical line $x = -\frac{b}{2a}$, which in algebraic terms says that $P\left(-\frac{b}{a} - x\right) = P(x)$, and this line is called the *axis* of the parabola. The point where the axis meets the parabola, which has coordinates $x = -\dfrac{b}{2a}$ and $y = P\left(-\dfrac{b}{2a}\right) = \dfrac{\Delta}{4a^2}$, is called the vertex of the parabola. If $a > 0$, then the vertex gives the minimum value of $P(x)$ over all x and if $a < 0$ then it is the maximum. The y-intercept is obtained at $x = 0$ and $y = P(0) = c$.

Assume that $P(x) = ax^2 + bx + c$ has two roots x_1, x_2. Then, $P(x)$ can be rewritten as

$$P(x) = a(x - x_1)(x - x_2) = ax^2 - a(x_1 + x_2)x + ax_1x_2.$$

Since two polynomials are equal when corresponding coefficients are equal, we have

$$x_1 + x_2 = -\frac{b}{a}, \qquad x_1x_2 = \frac{c}{a}. \tag{2}$$

These relations are called *Vieta's formulas* for quadratic polynomials. Note that knowing the coefficients of a quadratic equation, we can immediately find the sum and the product of its roots.

Example 1. Let a and b be real numbers such that $a + b \geq 8$. Prove that at least one of the quadratic equations $x^2 + ax + b = 0$ and $x^2 + bx + a = 0$ has real roots.

Adrian Andreescu

Solution. Assume by way of contradiction that $a^2 - 4b < 0$ and $b^2 - 4a < 0$. Then a and b are positive and

$$a^4 < (4b)^2 < 16(4a) = 64a,$$

implying $a < 4$. Similarly, $b < 4$, yielding $a + b < 8$, a contradiction. Hence the conclusion follows. ■

Example 2. Consider the numbers $a, b, c \in \mathbb{Z}$ and the sets

$$\begin{aligned} A &= \{x \in \mathbb{R} \mid x^2 + bx + c = 0\} \\ B &= \{x \in \mathbb{R} \mid x^2 + cx + a = 0\} \\ C &= \{x \in \mathbb{R} \mid x^2 + ax + b = 0\}. \end{aligned}$$

Prove that $A \cup B \cup C = \emptyset \implies a = b = c$.

Titu Andreescu - Gazeta Matematică Contest 1984

Solution. If $A \cup B \cup C = \emptyset$, then the equations $x^2+bx+c=0$, $x^2+cx+a=0$ and $x^2+ax+b=0$ have no real solutions. Since $a, b, c \in \mathbb{Z}$ and for all integers m and n we have $m^2 - 4n \equiv 0 \pmod 4$ or $m^2 - 4n \equiv 1 \pmod 4$, we get

$$b^2 - 4c \leq -3, \quad c^2 - 4a \leq -3, \quad a^2 - 4b \leq -3.$$

Adding side by side these inequalities, we get

$$(a-2)^2 + (b-2)^2 + (c-2)^2 \leq 3.$$

From this, we obtain that $a, b, c \in \{1, 2, 3\}$. Assume without loss of generality that $a = \min(a, b, c)$.

(i) If $a = 1$, then $c^2 \leq 4a - 3 = 1$ and $1 \leq c$. So, $c = 1$.

(ii) If $a = 2$, then $c^2 \leq 4a - 3 = 5$ and $2 \leq c$. So, $c = 2$.

(iii) If $a = 3$, then $c^2 \leq 4a - 3 = 9$ and $3 \leq c$. So, $c = 3$.

Thus we conclude that $c = a = \min(a, b, c)$. Hence repeating this argument with c in the place of a, we conclude that $a = c = b$. ■

Example 3. Determine the pairs of natural numbers (a, b) such that the equations $x^2 + ax + b = 0$ and $x^2 + bx + a = 0$ both have rational roots.

Titu Andreescu - Romanian Mathematical Olympiad 1985

Solution. If the two equations have rational roots, then their discriminants are perfect squares, i.e., $a^2 - 4b$ and $b^2 - 4a$ are both perfect squares. Assume without loss of generality that $a \geq b$. If $b = 0$, then $-4a \leq 0$ must be a perfect square, so $a = 0$. Let $b > 0$. If $a = b$, then $a^2 - 4a$ must be a perfect square. Since $a^2 - 4a \geq 0$, then $a \geq 4$. If $a \geq 5$, then

$$(a-3)^2 < a^2 - 4a < (a-2)^2$$

and so $a^2 - 4a$ is not a perfect square. If $a = 4$, then $a^2 - 4a = 0$, which is a perfect square. Now, assume that $a > b$, i.e., $a - 1 \geq b$. It follows that

$$(a-2)^2 \leq a^2 - 4b < a^2.$$

Thus, $a^2 - 4b = (a-1)^2$, i.e., $-4b = -2a + 1$, contradiction, or $a^2 - 4b = (a-2)^2$, i.e., $b = a - 1$. Then,

$$b^2 - 4a = (a-1)^2 - 4a = a^2 - 6a + 1$$

must be a perfect square. Since $a^2-6a+1=a(a-6)+1\geq 0$, then $a\geq 6$. If $a\geq 8$, then

$$(a-4)^2<a^2-6a+1<(a-3)^2$$

and so a^2-6a+1 is not a perfect square. If $a=6$, we get $a^2-6a+1=1$, a perfect square. If $a=7$, we get $a^2-6a+1=8$, which is not a perfect square. So, $(a,b)\in\{(0,0),(4,4),(6,5),(5,6)\}$. ■

Example 4. Given nonzero real numbers a, b, and c such that the quadratic equations (in x) $ax^2+bx+c=0$, $bx^2+cx+a=0$, $cx^2+ax+b=0$ share a common root, find all possible values of $\dfrac{a^2}{bc}+\dfrac{b^2}{ca}+\dfrac{c^2}{ab}$.

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Solution. Let α be the common root. Then

$$a\alpha^2+b\alpha+c=0,\qquad b\alpha^2+c\alpha+a=0,\qquad c\alpha^2+a\alpha+b=0.$$

Adding these three equations yields

$$(a+b+c)(\alpha^2+\alpha+1)=0.$$

If $\alpha^2+\alpha+1\neq 0$, then $a+b+c=0$. Then by the identity

$$a^3+b^3+c^3-3abc=(a+b+c)(a^2+b^2+c^2-ab-bc-ca),$$

we have $a^3+b^3+c^3=3abc$, and $\dfrac{a^2}{bc}+\dfrac{b^2}{ca}+\dfrac{c^2}{ab}=\dfrac{a^3+b^3+c^3}{abc}=3$.

If $\alpha^2+\alpha+1=0$, then $\alpha=\dfrac{-1\pm i\sqrt{3}}{2}$. But since a, b, c are real, it follows that $\overline{\alpha}$ is also a root of each of the original quadratic equations and so $a=b=c$, and the desired value is again 3. ■

Problems

Problem 1. Let $f(x)=ax^2-bx+c$, where a, b, c are primes. Given that the equation $f(f(x))=2023$ has roots 0, α, β, γ, with $\alpha+\beta+\gamma=\dfrac{26}{43}$, find a, b, c.

Problem 2. Let $a,b,c\in\mathbb{R}^*$ be pairwise distinct and let

$$\begin{aligned}A&=\{x\in\mathbb{R}\mid ax^2+bx+c=0\}\\B&=\{x\in\mathbb{R}\mid bx^2+cx+a=0\}\\C&=\{x\in\mathbb{R}\mid cx^2+ax+b=0\}.\end{aligned}$$

Prove that if $A\cap B\cap C\neq\emptyset$, then $A\cup B\cup C$ has exactly 4 elements.

Problem 3. Find the greatest negative integer n for which the equation

$$x^2 + nx + 2016 = 0$$

has integer solutions.

Problem 4. Let $a = \sqrt{r-1} + \sqrt{2r} + \sqrt{r+1}$ and $b = \sqrt{r-1} - \sqrt{2r} + \sqrt{r+1}$, where r is a real number greater than 1. Given that a and b are roots of the quadratic equation $x^2 + cx + \sqrt{2021} = 0$, find c.

Problem 5. Prove that for each integer $a \neq 0, 2$ there is a nonzero integer b such that the equation

$$ax^2 - (a^2 + b)x + b = 0$$

has integer roots.

Problem 6. Find all pairs (m, n) of nonzero integers such that $m + n$ is a root of the equation $x^2 + mx + n = 0$.

Problem 7. Find all pairs (a, b) of real numbers such that the roots of the equations $x^2 + ax - b = 0$ and $x^2 + bx - a = 0$ are 1, 2, 3, and 5 in some order.

Problem 8. Find all pairs of integers (m, n) such that both the equations $x^2 + mx - n = 0$ and $x^2 + nx - m = 0$ have integer roots.

Problem 9. Prove that there are more than 1985 triples (a, b, c) of nonzero integers a, b, c with $|a|, |b|, |c| \leq 27$ such that each of the equations

$$\begin{aligned} ax^2 + bx + c &= 0, \\ bx^2 + cx + a &= 0, \\ cx^2 + ax + b &= 0 \end{aligned}$$

has rational roots.

Problem 10. Let a, b, c be nonzero integers such that the equation

$$ax^2 + bx + c = 0$$

has rational roots. Prove that $b^2 \leq (ac + 1)^2$.

Problem 11. Let a, b, c be positive integers such that $b > a^2 + c^2$. Prove that the roots of the equation $ax^2 + bx + c = 0$ are real and irrational.