Contents

Preface to the English Edition

The book we are proposing here to the English speaking reader is a book that would have been classified at the beginning of the previous century as a book of "Modern Geometry" of the triangle and quadrilateral. As most of the results have been obtained in the second half of the 19th century and the first half of the 20th century, the term "modern" doesn't necessarily fit in the picture any longer. A better term would probably be "Advanced Euclidean Geometry", which is very much in use nowadays.

The author of this book was a retired artillery colonel and an enthusiastic amateur mathematician. This should come as no surprise, as for any artillery officer, mathematics (and especially geometry) plays an important part in his formation. The most famous example is, of course, that of Napoleon Bonaparte, whose name remained in the history of geometry because of the Napolean Theorem concerning equilateral triangles.

As the title surely suggests, this book is a rich collection of some of the most important properties of numerous points, lines, and circles related to triangles and quadrilaterals, as they were known by the mid 1950s. It is impossible to list all the subjects touched in a few lines here: the nine-point circle, Simson line, orthopolar triangles, orthopole, Gergonne and Nagel points, Miquel point and circle, Carnot circle, Brocard points, Lemoine point and circles, and Newton-Gauss line, are a few. It was probably one of the most complete descriptions of the subject at the moment of the writing.

Mihalescu relied heavily on sources belonging to the French school of mathematics. The most cited source is the classical book *Exercices de Géométrie*, by Frère Gabriel-Marie (F.G.-M.). It is a pity that this book, probably one of the best books of elementary geometry ever written, is not as well-known today as it deserves to be, although it is again in print as an older edition at Jacques Gabay. Other important sources are the old French (or Belgian) elementary mathematics journals, such as Mathesis, Journal de Mathématiques Élémentaires, Nouvelles Annales Mathématiques. This is, of course, understandable, considering the important contributions of the French mathematicians, especially in the subject of triangle geometry, but also the fact that in Romania, during the period when Mihalescu was educated, (the beginning of the 20th century), French was the most widely-spoken foreign language. One of the goals the author had was to disseminate the results of Romanian mathematicians; therefore, many results come from the main Romanian journal of elementary mathematics, Gazeta Matematică, as well as from other Romanian journals and monographs. As a bonus, the book includes many results, which are hardly available elsewhere, some of them unknown to western readers.

We have to disagree with the author on several points. First of all, in our opinion, the book is far from being as elementary as the author claims, and some of the problems are not as accessible to beginners as one might think. Secondly, the second part of the book (Appendix A), although it contains a lot of material, still implicitly assumes that the reader has a certain level of sophistication, especially in projective geometry. As such, he or she should be aware that it would be very useful to have some knowledge in this field. We can only suspect that the author consulted French books like the series of Papelier [12], but these are quite hard to find. Enough projective geometry can be found in the books by Johnson [14] or Court [13], recently republished by Dover.

The style of the book is not very far from that of other books from that period, and we tried to make as few changes and comments as possible. The reader should be aware that this is a book written before the Bourbaki movement left its mark on modern mathematics; therefore, we might sometimes feel that the exposition does not have enough rigor, although the proofs are correct and complete in accordance with the standards of the period. We made no attempt to change that – it would have meant to rewrite the entire book. We made a single exception related to angles. More specifically, some of the proofs impose some choice related to the position of some points in the figure. In many cases, this choice is related to the measure of some angles. Normally, one should examine the situation separately for each choice. Fortunately, by using directed angles, it is possible to handle all possible situations at the same time. As such, we modified some of the proofs, introducing directed angles to handle all the situations (usually, the author made a choice and performed the proof based on that choice, without mentioning the other possibilities). Since this method of directed angles is not as well known as it should be (although, surprisingly, it is quite old), we felt the need to add a second appendix to the book (written by Paul Blaga), devoted to this subject.

Apart from that, as we said before, we kept the modifications to a minimum: we completed the indices, added some extra bibliography, quietly corrected obvious mistakes, and added a notation list, including some of the less well-known or non-standard notations, without any pretention of exhaustivity.

Last, but not least, we have to thank Richard Stong for helping us correct the English of this book and for the large number of corrections and suggestions which come as a great improvement. As always, the mistakes that are still here belong to us.

The Editors

Author's Preface

This book has not been written due to the accumulation of properties of elementary geometry discovered during my research in this field, but because of the need for a book which should contain, in a concentrated and connected manner, both the properties of the remarkable points, lines, and circles and their solutions using only pure geometry. I hope to bring a modest contribution to the progress of our country.

All solutions are given. It is, thus, easy, even for beginners, to work the given properties over. There is no link missing from the chain of properties a solution is based on. We have added, for this reason, the second part of the work where all properties preliminary to the ones from the first part are given. We have, thus, completely removed the mention "one knows that", frequently used by different publications when referring to a property from another article or paper.

One of the goals I tried to achieve was making geometry among the young students more popular. This is the reason why the solutions for the given theorems are as simple as possible, while their explanations are as complete and as clear as possible. A few indications, which mark the path that needs to be followed in the solution of a problem, may be enough for an advanced geometer. This may not be necessarily the case for most young students. Hence, the book is firstly addressed to the young students. I am sure that the problems, no matter how hard they may seem at first sight, will prove to be understandable.

This book will be of much help, I hope, for the solvers of the problems of elementary geometry, as well. They will find here sufficient properties to rely upon. Geometers will also find here new problems to inspire them.

All problems given in this work – both the statements and their solutions, except for the classical ones or the ones for whom the author's name is given – should be considered original. Sure, as our great mathematician and engineer I. Ionescu (1870-1946), said in his article *Errare humanum est* published in Gazeta Matematică, vol. XLVII, 1941, p. 393, "it is possible for several researchers to obtain the same results almost at the same time or for someone to rediscover what had been discovered by others before".

Considering different particular cases may extend many problems from the work. I hope readers will be interested in working them over.

I tried not to make this work too large in order not to laden its usage for the solvers of geometry problems, for the researchers, or for the young students interested in enlarging their knowledge regarding elementary geometry. This work should be extended afterwards periodically, both for the enlargement of the properties from its chapters and for the adding of new chapters referring to other remarkable points, lines, circles or conics. Hence, this volume, which is just a start, does not stop the research in the field of the remarkable elements in geometry. I hope that a future edition of this book will add some of the original works given by the readers and containing remarks upon this work.

I thank General Gh. I. Popescu and engineer N. Blaha for their advice regarding this book and their elegant solutions using pure geometry given to some problems computationally solved by others before. I also thank the staff from superior education who was kind enough to read and appreciate this work.

In the end, I feel obliged to add the fact that this volume proves the big contribution brought by the Romanian mathematicians to the field of geometry.

Bucharest, February 1955
CONSTANTIN MIHALESCU
artillery colonel in retirement

CHAPTER 1

The Nine-Point Circle (Euler Circle), Euler Line and Feuerbach Points

1.1. *In a triangle, the midpoints of the sides, the feet of the altitudes and the midpoints of the segments intercepted between the vertices and the orthocenter*[1]*, lie on a circle* (the nine-point circle or the Euler circle).

In the triangle ABC (figure 1), let A', B', C' be the midpoints of the sides, A_1, B_1, C_1 the feet of the altitudes, H the orthocenter and A_2, B_2, C_2 the midpoints of the segments AH, BH, CH (*the Eulerian points*). We will show that these nine points are concyclic.

Indeed, as the points A_2, C_2 are the midpoints of the segments AH, CH, we have that the line A_2C_2 is parallel to AC. Because A' is the midpoint of the side BC, the line C_2A' is parallel to BH. Since, the altitude BH of the triangle makes a right angle with the side AC, it follows that A_2C_2 and C_2A' are perpendicular to each other. In the same way, the angle $\angle A_2B_2A'$ is of ninety degrees. Thus, the right-angled triangles A_2C_2A' and A_2B_2A' are inscribed in the same circle which has as a diameter the common hypotenuse A_2A'. But the point A_1 lies on this circle as well because $\angle A_2A_1A' = 90°$.

Analogously, we can prove that the points B', B_1 and C', C_1 lie on the same circle as A_2, B_2, C_2. So the theorem is proven.

1.2. *Remark 1.* Because the segments are chords in the nine-point circle (O_9), we obtain the following property:

[1] The common point of the altitudes of the triangle

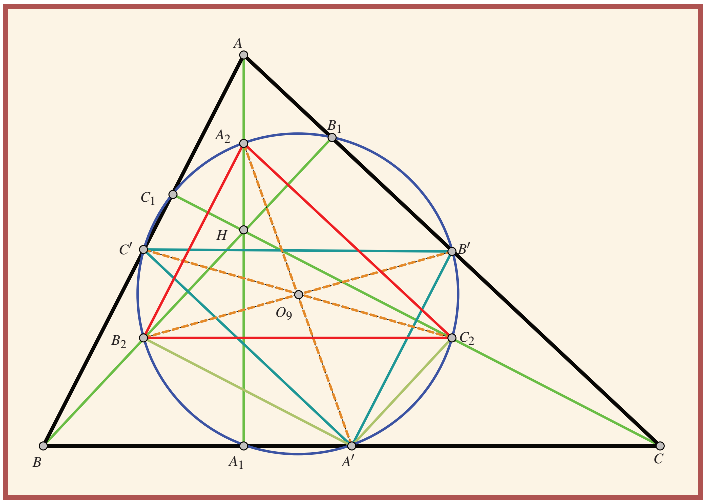

Figure 1

The perpendiculars to the sides of a triangle at the midpoints of the segments between the feet of the altitudes and the midpoints of these sides, are concurrent in the center of the nine-point circle.

1.3. *Remark 2*. Because the lines $A'A_2, B'B_2, C'C_2$ are diameters in the circle (O_9), we may state:

In a triangle, the lines which join the midpoints of the sides, respectively, with the Eulerian points of the altitudes from the opposite vertices, are diameters in the nine-point circle.

1.4. *Remark 3*. As the points A' and A_2, B' and B_2, C' and C_2 are diametrically opposed in the circle (O_9), one obtains the following property:

In a triangle, the triangle whose vertices are the Eulerian points and the median triangle[a] are symmetric[b] and inversely homothetic[c], having the center of symmetry and the center of homothety in the center of the nine-point circle.

[a]The *median triangle* of a triangle T is the triangle which has its vertices at the midpoints of the sides of the triangle T. The median triangle is also called complementary triangle.

[b]The points A and A' are called *symmetric* with respect to the point O (called the *center of symmetry*) if O is the midpoint of the segment AA'. The points A and A' are called *symmetric* with respect to the line XY (called *axis of symmetry*) if this line is the perpendicular bisector of the segment AA'. Two figures $\mathscr{F}$ and $\mathscr{F}'$ are called *symmetric* with respect to a given point or axis, if the points of these figures are symmetric, two by two, with respect to the considered point or axis.

[c]We join a point O from the plane of a figure $\mathscr{F}$ with different points $A, B, C, \ldots$ on the figure. We take on the lines $OA, OB, OC, \ldots$ the directed segments $\overline{OA'}, \overline{OB'}, \overline{OC'}, \ldots$ such that:

$$\frac{\overline{OA'}}{\overline{OA}} = \frac{\overline{OB'}}{\overline{OB}} = \frac{\overline{OC'}}{\overline{OC}} = \ldots = k.$$

Then, we obtain a figure $\mathscr{F}'$ called *homothetic* to the figure $\mathscr{F}$. The constant k is called *ratio of homothety*, while the point O is called *center of homothety*. The points A and A', B and B', C and C' etc, are called *homologous points*, the directed segments $\overline{OA}$ and $\overline{OA'}$, $\overline{OB}$ and $\overline{OB'}$, $\overline{OC}$ and $\overline{OC'}$ etc, which join two homologous points to the homothety center are called *homologous radii*; the lines AB, $A'B'$ etc, which join in the homothetic figures $\mathscr{F}$ and $\mathscr{F}'$ the homologous points, are called *homologous lines*. If the homologous radii are heading in the same direction, we say that the figures are *directly homothetic*; if they point in opposite directions, we say that the figures are *inversely homothetic*.

1.5. *Remark 4.* Because the triangle ABC is the anticomplementary triangle[1] of $A'B'C'$, we have the following property:

The circumcircle of a given triangle is the nine-point circle of the anticomplementary triangle. (The vertices of any triangle are the midpoints of the sides of the anticomplementary triangle).

1.6. *The center of the nine-point circle lies on the line which joins the orthocenter to the circumcenter and divides the distance between these two points in two equal parts. (the Euler line).*

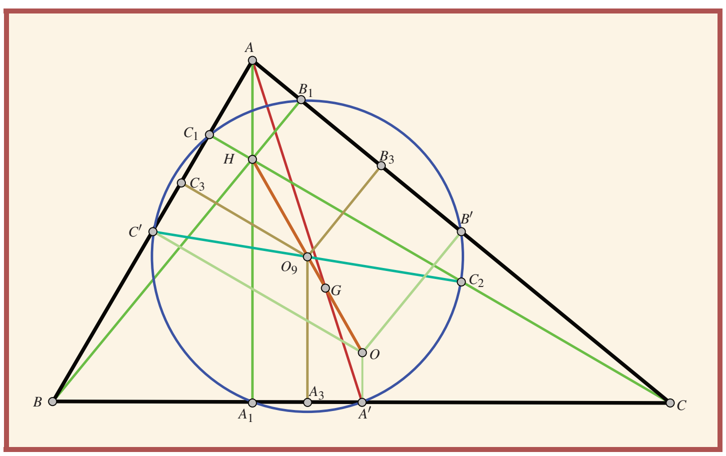

Figure 2

[1]The anticomplementary triangle of an arbitrary triangle T has its sides parallel to the sides of T passing through the vertices of T.

In the triangle ABC (figure 2), let A_1, B_1, C_1 and A', B', C' be the feet of the altitudes and the midpoints of the sides BC, CA, AB respectively. Let H be the orthocenter, O and O_9 the circumcenter and the center of the nine-point circle respectively. Let A_3, B_3, C_3 be the midpoints of the segments A_1A', B_1B', C_1C'.

As the altitude AA_1 and the perpendicular bisector $A'O$ are parallel, both being perpendicular on the same line BC, the quadrilateral A_1HOA' is a trapezoid. So the perpendicular raised at A_3, the midpoint of the side A_1A', passes through the midpoint of the segment OH. Analogously, the perpendiculars raised on CA and AB at the midpoints of the sides B_1B' and C_1C' of the trapezoids B_1HOB' and C_1HOC' pass through the midpoint of their common side OH. But these perpendiculars, raised on the sides of the triangle ABC at the points A_3, B_3, C_3 are concurrent in the center on the nine-point circle (see **1.2**). As they also pass through the midpoint of OH, it follows that the center O_9 lies at the midpoint of the segment OH.

Another proof: Let C_2 be the midpoint of the segment CH. The altitude CC_1 and the perpendicular bisector of the side OC' are parallel, while the segments C_2H and OC' are equal (see **A.1**). So, the segments C_2C' and HO have a common point O' which divides them in two equal parts, so $C_2O' = O'C'$ and $HO' = O'O$. Yet, C_2C' is a diameter of the circle (O_9) (see **1.3**). Hence, we obtain that O' is in fact O_9.

1.7. *Remark 1.* As $OA' = AH/2$ (see **A.1**), the line AA', a median of the triangle ABC and the Euler line HO, are intersecting each other at a point G. Hence $GA' = AG/2$ and $GO = HG/2$. The first equality shows that the point G is the centroid[1] of the triangle ABC, while the second one gives the position of G on the Euler line. Thus, we have the following property:

The centroid G of the triangle ABC lies on Euler line HO and

$$GO = \frac{1}{2}HG.$$

1.8. *Remark 2.* The theorem **1.6** and the remark from above give the following relation between the points H, O_9, G and O lying on the Euler line, with respect to the smallest segment GO_9:

$$GO_9 = \frac{GO}{2} = \frac{OO_9}{3} = \frac{HG}{4} = \frac{HO}{6}.$$

1.9. *On the Euler line, the points H, O_9, G and O (the orthocenter, the center of the nine-point circle, the centroid and the circumcenter) form an harmonic division.*

These four points lying on a line, form an harmonic division if we have the following relation:

$$\frac{HO_9}{HO} = \frac{GO_9}{GO} \tag{1}$$

[1] The common point of the medians of a triangle.

The relations between the segments of this equality are:

$$HO_9 = \frac{1}{2}HO; \quad GO_9 = \frac{1}{6}HO \text{ and } GO = \frac{1}{3}HO,$$

whence:

$$\frac{HO_9}{HO} = \frac{1}{2}\frac{HO}{HO} = \frac{1}{2}$$

and

$$\frac{GO_9}{GO} = \frac{\frac{1}{6}HO}{\frac{1}{3}HO} = \frac{1}{2}.$$

Because the equality (1) is verified, the points H, O_9, G and O form an harmonic division.

1.10. *The radius of the nine-point circle is equal to half of the radius of the circumcircle.*

In the triangle ABC (figure 3), let O and O_9 be the circumcenter and the center of the nine-point circle respectively. Let H be the orthocenter and B_2, the Eulerian point on the altitude BB_1.

Because in the triangles HBO and HB_2O_9 we have $HB = 2HB_2$ and $HO = 2HO_9$ (see **1.6**), we also get $BO = 2B_2O_9$. But B_2O_9 is a radius of the nine-point circle, while BO is a radius of the circumcircle, so the proposition is proven.

Another solution. The triangle ABC and its median triangle are similar, having the ratio of the sides $1/2$. Hence, we obtain that the radii of their circumcircles are in the same fraction.

1.11. *Remark.* As a consequence of the theorem from above, we obtain the following property:

> All triangles having the same circumcircle, have congruent nine-point circles.

1.12. *Consider a triangle ABC with circumcircle (O). The circles having as diameters the radii AO, BO, CO, are symmetric to the nine-point circle with respect to the corresponding sides of the median triangle.*

As OA' and OB' are perpendicular bisectors of the sides BC and AC (figure 3), we get

$$\angle OA'C = \angle OB'C = 90^\circ.$$

Therefore the circle having OC as a diameter passes through the points A' and B'. These two points also belong to the circle (O_9). It follows that the side $A'B'$ of the median triangle is a common chord for these circles.

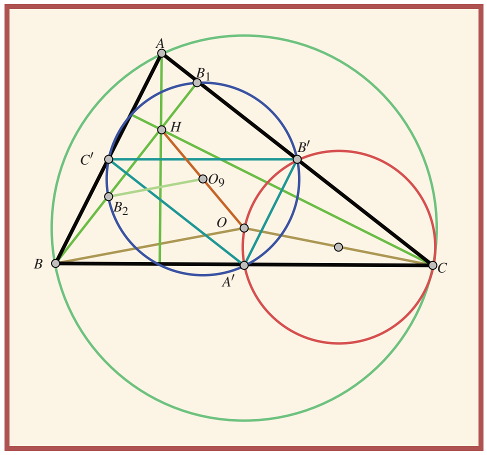

Figure 3

Because the radius of the circumcircle is equal to the diameter of the circle (O_9) (see **1.10**), the circle described by OC as a diameter will be congruent to the circle (O_9).

These circles are congruent and have the side $A'B'$ as a common chord, so they will be symmetric with respect to this side.

1.13. *Remark.* The following result follows immediately, from the previous theorem:

> In a given triangle, the circles symmetric to the nine-point circle with respect to the sides of the median triangle, are tangent to the circumcircle at the vertices of the given triangle.

1.14. *The nine-point circle and the circumcircle of the given triangle have the center of direct homothety at the orthocenter. The center of inverse homothety lies at the centroid.*

The sides of the triangle ABC are parallel with those of the triangles formed by the Eulerian points $A_2B_2C_2$ and with those of the median triangle $A'B'C'$. This parallelism shows that these triangles are homothetic.

The triangle ABC and the triangle $A_2B_2C_2$ have their altitudes in common, while the triangle ABC and the triangle $A'B'C'$ have common medians (figure 4). So it follows that the homothety centers are the orthocenter H and the centroid G, respectively.

The homothety of the triangles ABC and $A_2B_2C_2$ is direct, because the homologous segments HA and HA_2 have the same direction. The homothety of the triangles ABC

and $A'B'C'$ is inverse because the homologous segments GA and GA' have opposite directions.

So the circumcircles of the triangles ABC, $A_2B_2C_2$ and $A'B'C'$, the latter being the circle O_9 of ABC, have the center of direct homothety in H and the center of inverse homothety in G.

1.15. *Remark 1*. The nine-point circle divides in two equal parts the segments starting at the orthocenter and ending at the circumcircle.

From the orthocenter H we draw a line which cuts the circle (O_9) in E and the circle (O) in F (figure 4). Because these circles have the homothety ratio $1/2$ and the center of direct homothety in H, the ratio of the homologous segments will be $HE/HF = 1/2$. So $HE = EF$.

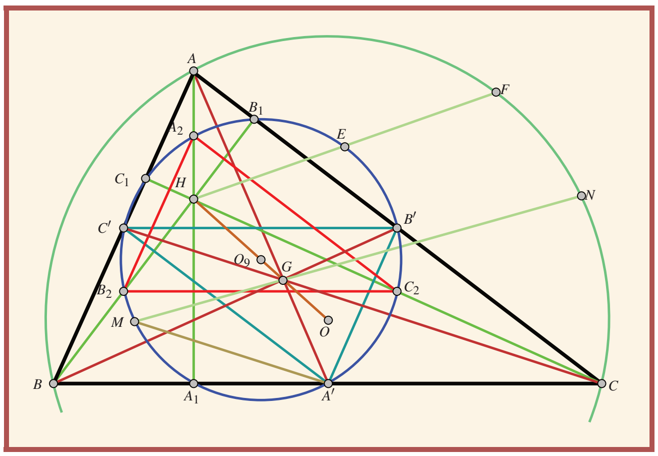

Figure 4

1.16. *Remark 2*. The distance from some point M on the nine-point circle to the centroid G of the given triangle ABC is equal to half of the distance from the centroid G to the point where the extension of the segment MG cuts the circumcircle (figure 4).

Let N be the point where the prolongation of the segment MG intersects the circumcircle. Because the circles (O_9) and (O) have the inverse homothety center in G, the ratio of the homologous segments GM and GN equals to the ratio of homothety of the circles. As this ratio is $1/2$, we get $GM = GN/2$.

1.17. *Remark 3*. In the inverse homothetic triangles ABC and $A'B'C'$ (figure 4), the points A and N are homologous to the points A' and M, respectively. So, the lines AN and $A'M$ are parallel.

Having in mind that the homothety center of these triangles lies at G, we may state the following property:

One draws from the vertices A,B,C, of any triangle the chords of the circumcircle AA_3, BB_3, CC_3 parallel to an arbitrary direction. One also draws from the midpoints A', B', C' of the sides BC, CA, AB, in the nine-point circle, the chords $A'A_4, B'B_4, C'C_4$, parallel to the same direction. Then the lines A_3A_4, B_3B_4, C_3C_4 will be concurrent in the centroid of the given triangle ABC (figure 5).

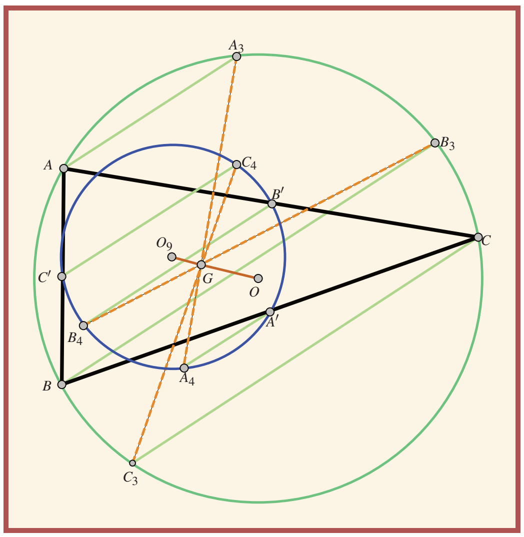

Figure 5

1.18. *The diameters of the nine-point circle which pass through the midpoints of the sides of the given triangle, are parallel to the radii of the circumcircles which pass through the opposite vertices of the considered sides.*

In the triangle ABC (figure 6) with circumcircle (O), the diameter of the nine-point circle (O_9) which passes through the midpoint A' of the side BC, also passes through the Eulerian point A_2 of the altitude AA_1. But $AA_2 \| OA'$ and $AA_2 = OA'$ (see **A.1**). Hence, the quadrilateral $AA_2A'O$ having the opposite sides AA_2 and OA' is a parallelogram, so $A'A_2 \| AO$.

1.19. *Remark 1.* The radius OA (figure 6) is perpendicular to the tangent AT to the circumcircle (O) of ABC and parallel to the diameter $A'A_2$ which is perpendicular to the tangents of the circle (O_9) passing through A' and A_2. So, these last two tangents

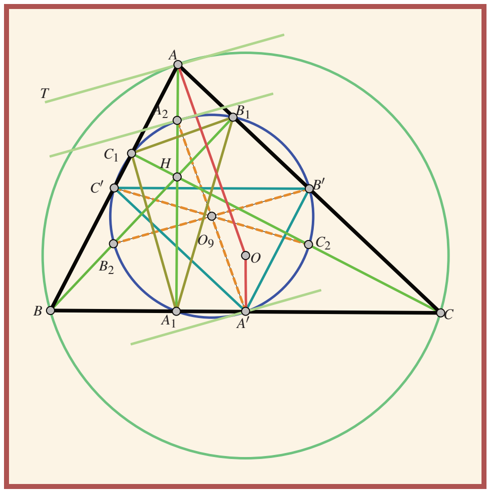

Figure 6

passing through A' and A_2 and the first one, AT, are parallel to each other. (The parallelism of these three tangents also follows from the homothety of the triangles ABC, $A'B'C'$ and $A_2B_2C_2$ with the homologous vertices A, A' and A_2). On the other hand, the tangent AT is parallel to the side B_1C_1 of the orthic triangle[1] because these lines are antiparallel[2] to the side BC with respect to the angle $\angle A$ ($\angle TAB = \angle B$, having the same measure and $\angle B = \angle AB_1C_1$, the quadrilateral BC_1B_1C being cyclic). It follows that the tangent AT to the circle (O), the tangents at A' and A_2 to the circle (O_9) and the side B_1C_1 of the orthic triangle are parallel to each other. Thus, we may state the theorem:

In a triangle, the tangents of the nine-point circle which pass through the midpoint of one side and through the Eulerian point of the altitude which falls on this side, respectively, are parallel to the tangent to the circumcircle at the opposite vertex of the considered side and to the corresponding side of the orthic triangle, respectively.

1.20. *Remark 2*. From the theorems from above we may also conclude:

[1]The orthic triangle of a triangle T is the triangle which has as vertices the feet of the altitudes of the triangle T.

[2]Two lines AB and CD are antiparallel with respect to the angle O, formed between the lines AC and BD, if $\angle OAB = \angle ODC$ (or $\angle OBA = \angle OCD$)

a) The tangents of the nine-point circle passing through the midpoints of the sides are antiparallel to these sides with respect to the opposite angles of the given triangle.

b) The diameters of the nine-point circle which pass through the Eulerian points are the perpendicular bisectors of side of the orthic triangle.

c) The diameters of the nine-point circle which pass through the midpoints of the sides are perpendicular on the corresponding sides of the orthic triangle and fall at their midpoints.

d) The midpoint of one side of the given triangle is the vertex of the isosceles triangle which has as a base the opposite side of the orthic triangle.

e) The perpendiculars passing through the midpoints of the sides of the given triangle, respectively to the sides of the orthic triangle, are concurrent in the nine-point center of the given triangle.

1.21. *Remark 3.* The points A' and A_2 are the midpoint of the side BC and the midpoint of the segment AH, respectively, which are diagonals in the complete quadrilateral[1] BC_1B_1C. So, it follows that (see **9.5**) the diameter $A'A_2$ is the Newton-Gauss line[2] of this quadrilateral. This means:

In any triangle, the diameters of the nine-point circle which passes through the midpoints of the sides, are the Newton-Gauss lines of the complete quadrilaterals formed by two sides and the altitudes which fall on these sides.

Editors' Note. Sometimes, a complete quadrilateral is defined by specifying a triangle ABC and a transversal Δ of the triangle. The other three vertices of the complete quadrilateral are the intersections between the transversal and the sides of the triangle. Therefore, we shall use the notation (ABC, Δ) to denote the corresponding quadrilateral.

1.22. *The circumcircle of a triangle is the nine-point circle of the antisupplementary triangle*[3].

This property follows from the fact that the vertices of the given triangle are the feet of the altitudes of the antisupplementary triangle (see **A.14**).

---

[1] A *complete quadrilateral* is the figure determined by four lines, no three of which are concurrent, and their six points of intersection. The (Editors' Note).

[2] The Newton-Gauss line of a complete quadrilateral is the line passing through the midpoints of the three diagonals of the quadrilateral (see chapter 9).

[3] The *antisupplementary triangle* of a triangle T is the triangle formed by the exterior angle bisectors of this triangle T.