Introduction

Functional equations have been of significant importance in mathematics for the last couple of centuries, as well as a constant presence at high-level mathematical competitions. Historically, in mathematical research, functional equations have been closely associated with analysis — some of the greatest mathematicians have dedicated themselves to solving them. In present-day research, certain differential equations are of central importance in analysis, differential geometry, and other various areas of theoretical physics. Because of their importance and popularity, differential equations are thoroughly studied, and there is an established theory of classical differential equations, taught in universities, as well as a multitude of resources which present more recent advances in the field.

However, this is not the case for functional equations that are given in high-school or even university contests. Due to the constraints of maintaining an elementary threshold, they have evolved along a quite different route, in that there is a scarcity of math Olympiad textbooks that implement a systematic approach to such equations. Few books accessible to high-school students move beyond discussing the ubiquitous Cauchy's Equations and their slight generalizations. If one were to inquire about a strong comprehensive mathematical Olympiad book about functional equations, Venkatachala's fine book [24] and Small's book [23] come to mind. There are several good articles and notes on this topic scattered throughout the literature, but we have not encountered any book that offers a comprehensive study of this subject.

We deem that functional equations are severely underestimated by competitive mathematical literature. This subject occupies a special position in current mathematical contests. Functional equations are consistently featured in competitions, especially at the International Mathematical Olympiad, but it also appears that they are usually not considered as a distinct subject, but rather as a subset of algebra — a collection of problems and a collection of tricks to solve them.

We think that functional equations should be treated as a separate topic instead. An experienced competitor will say that every area of mathematics has its own distinct 'flavor', a different mindset and approach to handling its problems. This is why some are better at synthetic geometry than inequalities, or more skilled at number theory than combinatorics, despite equal time devoted to each of these topics. The authors, having been strongly involved in mathematical Olympiads for many years, both as competitors and as coaches, are certain that functional equations have a 'feel' of their own. To solve advanced functional equations, one needs to have a deep aptitude for algebraic manipulations, creativity for noticing clever substitutions, and a wide experience in all areas of mathematics from which functional equations' ideas stem. The way one searches for a solution for a functional equation can be quite different from the way one tries to solve a geometry problem or an inequality. Thus, functional equations belong to a field of their own.

It is known that mathematical competitions are a very efficient way to learn and hone problem-solving skills. These skills are later essential in several high-powered quantitative careers, including research in mathematics, physics, or computer science; or more applied work in engineering, finance, or computer programming. Functional equations, as part of problem solving, play their own important role in teaching the mind how to think. If one were to compare training the brain to training the body, functional equations would be like a set of exercises targeting certain muscle groups (but involving all other muscles to some extent as well).

This book is a systematic and comprehensive approach to functional equations as a whole. Unlike in other branches of competitive mathematics, there is very little theory – rather, the methods and techniques utilized in solving these equations play the most important part. For this reason, the book has taken a highly practical path, including lots of problems designed to teach students how to familiarize themselves with every strategy employed, as well as to experiment in combining and manipulating different techniques.

This work contains all the important functional equations given at contests in recent years, classified by the way they are solved. It explains the reasoning behind each method, and offers advice on how to invent a meaningful solution. We are confident that talented and motivated students who work through this book will significantly expand their horizons and capabilities in solving functional equations given in any mathematical context. And, in the meantime, they will have acquired a certain way of thinking about problems, which will not only be useful in other areas of mathematics, but later in life as well.

The book begins with the most popular functional equation — Cauchy's equation. Several versions and extensions are presented. The chapter is also enhanced with a lot of examples and exercises designed to aid readers in fully absorbing this important equation, and learn how to apply it in other contexts. Two other chapters focus on key functional equations, but the rest of the chapters are centered around methods rather than specific equations. Some chapters touch on issues not mentioned anywhere else in this type of literature. For example, the chapter on approximating by linear functions or the subsection on polynomial recurrences and continuity demonstrate the uniqueness of this work. Because functional equations are so diverse, many of these chapters are independent from one another and could be read in a different order than as presented here. A reader with a specific goal in mind can proceed directly to a chapter they desire. Some chapters are shorter than others because there are much fewer problems illustrating the corresponding methods presented — we strove for efficiency rather than artificial homogeneity in chapter size. The second to last chapter of the book contains a list of problems labeled "miscellaneous"; these are questions that were hard to categorize, although they are still grouped in subsections of related problems. This is intended to be the final chapter one reads; a lot of problems that are very challenging. We advise the reader not to tackle these questions before they are comfortable with the methods employed in the rest of the book. The final chapter contains the solutions to all the problems in the book. Though it is recommended that each question may be attempted, we recognize that you can sometimes learn more from reading the solution to a hard problem rather than producing a solution to an easier problem by yourself.

We would like to thank our dear friend Gabriel Dospinescu for the problems suggested, as well as for his pertinent remarks.

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About the second edition

This edition includes 22 more problems from recent mathematical contests such as national or regional Olympiads, team selection tests, and IMO. Some solutions to problems in the first edition have been revised and several typos have been corrected.

We would like to thank Lyuben Lichev for his useful observations.

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Third edition

This is a revision of the second edition. Several solutions to problems have been improved. A number of typos and typographical infelicities have been corrected.

November, 2019

Contents

Chapter 1

Cauchy's Equations

1.1 Additive Cauchy Equation

Let D be a set of real numbers such that $x + y \in D$ for all x, $y \in D$.
A function $f : D \to \mathbb{R}$ is called additive if

$$f(x + y) = f(x) + f(y) \tag{1.1}$$

for all $x, y \in D$. The simplest examples of functions satisfying (1.1) are the linear functions of the form $f(x) = ax$, where $a \in \mathbb{R}$. For $D = \mathbb{R}$ the functional equation (1.1) has been first studied systematically by Cauchy [8] and it is now known as the additive Cauchy equation. It is natural to ask if the linear functions $f(x) = ax$ are the only solutions of the additive Cauchy equation. As we shall see later the answer to this question is *No* in general and *Yes* if we impose some additional assumptions on the functions under consideration. The next proposition is a direct consequence of (1.1).

Proposition 1.1. *Let D be a set of real numbers such that $0 \in D$, $x + y \in D$ and $\frac{x}{n} \in D$ for all $x, y \in D$ and all integers n. If $f : D \to \mathbb{R}$ is an additive function then*

$$f\left(\sum_{k=1}^{n} r_k x_k\right) = \sum_{k=1}^{n} r_k f(x_k)$$

for arbitrary $x_k \in D$ and rational numbers r_k, $1 \le k \le n$.

Proof. Setting $y = 0$ and $y = -x$ in (1.1) gives $f(0) = 0$ and $f(-x) = -f(x)$, $x \in D$. Moreover, it follows by induction on n that

$$f\left(\sum_{k=1}^{n} x_k\right) = \sum_{k=1}^{n} f(x_k). \tag{1.2}$$

In particular, $f(nx) = nf(x)$ for any $x \in D$ and $n \in \mathbb{N}$.
Let m and n be positive integers, $r = \frac{m}{n}$ and $x \in D$. Then $mx \in D$ and $\frac{mx}{n} \in D$, i.e. $rx \in D$. Hence $nf(rx) = f(nrx) = f(mx) = mf(x)$ and therefore $f(rx) = rf(x)$. This identity also holds true for any non-positive rational number r since $f(0) = 0$ and $f(-y) = -f(y)$ for all $y \in D$. Now taking into account (1.2) we get that

$$f\left(\sum_{k=1}^{n} r_k x_k\right) = \sum_{k=1}^{n} f(r_k x_k) = \sum_{k=1}^{n} r_k f(x_k). \qquad \square$$

Example 1.1. Each additive function $f : \mathbb{Q} \to \mathbb{R}$ has the form $f(x) = ax$, where $a \in \mathbb{R}$. Indeed, if $x \in \mathbb{Q}$ then $f(x) = xf(1) = ax$, where $a = f(1)$.

Example 1.2. Let $\mathbb{Q}\left(\sqrt{2}\right) = \left\{p + q\sqrt{2} \mid p, q \in \mathbb{Q}\right\}$. For any $x = p + q\sqrt{2}$ set $\overline{x} = p - q\sqrt{2}$. Then any additive function $f : \mathbb{Q}(\sqrt{2}) \to \mathbb{R}$ has the form $f(x) = ax + b\overline{x}$, where $a, b \in \mathbb{R}$. To see this let $x = p + q\sqrt{2}$. Then it follows from Proposition 1.1 that $f(x) = pf(1) + qf(\sqrt{2})$. On the other hand,

$$p = \frac{x + \overline{x}}{2} \text{ and } q = \frac{x - \overline{x}}{2\sqrt{2}}.$$

Hence

$$f(x) = \left(\frac{x + \overline{x}}{2}\right) f(1) + \left(\frac{x - \overline{x}}{2\sqrt{2}}\right) f(\sqrt{2})$$

and setting

$$a = \frac{f(1)}{2} + \frac{f(\sqrt{2})}{2\sqrt{2}}, \quad b = \frac{f(1)}{2} - \frac{f(\sqrt{2})}{2\sqrt{2}}$$

we get that $f(x) = ax + b\overline{x}$. Conversely, any function $f : \mathbb{Q}(\sqrt{2}) \to \mathbb{R}$ of this form is additive since

$$f(x + y) = a(x + y) + b(\overline{x + y}) = ax + b\overline{x} + ay + b\overline{y} = f(x) + f(y)$$

for all $x, y \in \mathbb{Q}(\sqrt{2})$.

In the next theorem we give some necessary and sufficient conditions for an additive function f to be of the form $f(x) = ax$.

Theorem 1.1. *Let $f : \mathbb{R} \to \mathbb{R}$ be an additive function. Then the following four conditions are equivalent:*
(i) f has the form $f(x) = ax$ for some $a \in \mathbb{R}$;
(ii) f is bounded above (below) on an interval;
(iii) f is increasing (decreasing) on an interval;
(iv) f is continuous at a point.

Proof. It is clear that the condition (i) implies the other three. We shall prove now that (ii) $\to$ (i), (iii) $\to$ (ii) and (iv) $\to$ (ii).
(ii) $\to$ (i). Set $g(x) = f(x)-f(1)x$. Then $g : \mathbb{R} \to \mathbb{R}$ is an additive function and it follows from Proposition 1.1 that $g(r) = 0$ for any $r \in \mathbb{Q}$. Since the function $f(x)$ is bounded above on an interval (p, q), it follows that the function $g(x)$ is also bounded above on this interval. Let $g(x) < C$ for any $x \in (p, q)$ and let x_0 be an arbitrary real number. The interval $(p - x_0, q - x_0)$ contains a rational number r. Since $r + x_0 \in (p, q)$ we get

$$g(x_0) = g(r + x_0) - g(r) = g(r + x_0) < C.$$

Hence $g(x) < C$ for any $x \in \mathbb{R}$. Then

$$g(x_0) = \frac{1}{n}g(nx_0) \leq \frac{C}{n} \text{ and } g(x_0) = -\frac{1}{n}g(-nx_0) \geq -\frac{C}{n}$$

for any $n \in \mathbb{N}$. Thus

$$-\frac{C}{n} \leq g(x_0) \leq \frac{C}{n}$$

for any $n \in \mathbb{N}$ which clearly implies that $g(x_0) = 0$. Since $x_0 \in \mathbb{R}$ is arbitrary we get that $f(x) - f(1)x = g(x) = 0$, i.e. $f(x) = ax$, where $a = f(1)$.
The case when the function $f(x)$ is bounded below on an interval is considered analogously.
(iii) $\to$ (ii). If the function f is increasing (decreasing) on an interval then f is bounded above (below) on a smaller interval.
(iv) $\to$ (ii). Let the function $f(x)$ be continuous at a point $x_0 \in \mathbb{R}$. Then there is $\delta > 0$ such that if $|x - x_0| < \delta$ then $|f(x) - f(x_0)| < 1$. Hence

$f(x) < 1 + f(x_0)$ ($f(x) > f(x_0) - 1$) for any $x \in (x_0 - \delta, x_0 + \delta)$, i.e. f is bounded above (below) on the interval $(x_0 - \delta, x_0 + \delta)$. □

Corollary 1.1. *Let $f : \mathbb{R} \to \mathbb{R}$ be an additive function which is also multiplicative, i.e. $f(x + y) = f(x) + f(y)$ and $f(xy) = f(x)f(y)$ for all $x, y \in \mathbb{R}$. Then either $f(x) \equiv 0$ or $f(x) \equiv x$.*

Proof. Since $f(x^2) = f(x)^2 \geq 0$ it follows that f is an additive function which is bounded below on the interval $(0, +\infty)$. Hence it follows from Theorem 1.1 that $f(x) = ax$, where a is the constant. Then

$$axy = f(xy) = f(x)f(y) = a^2xy$$

for any $x, y \in \mathbb{R}$. Hence $a = a^2$ and we get that $a = 0$ or $a = 1$. □

Theorem 1.1 shows that if an additive function $f : \mathbb{R} \to \mathbb{R}$ does not satisfy one of the conditions (i)–(iv) then it does not satisfy the other three. We shall show that such additive functions do exist and that they have very strange behaviour.

Theorem 1.2. *Let $f : \mathbb{R} \to \mathbb{R}$ be an additive function which does not have the form $f(x) = ax$. Then any square (circle) in the plane contains a point from the graph of f.*

Proof. We shall give two proofs of this theorem.
First Proof. It is enough to prove the theorem for a square with sides parallel to the coordinate axes. Denote by $P_0(x_0, y_0)$ the center of such a square and let $2\varepsilon > 0$ be the length of its sides. Then for any point $M(x, y)$ inside the square we have $x_0 - \varepsilon < x < x_0 + \varepsilon$ and $y_0 - \varepsilon < y < y_0 + \varepsilon$. It follows from Theorem 1.1 that the function $f(x)$ is unbounded above and below on the interval $(x_0 - \varepsilon, x_0 + \varepsilon)$. Hence there exist points x_1, $x_2 \in (x_0 - \varepsilon, x_0 + \varepsilon)$ such that $y_1 = f(x_1) < y_0$ and $y_2 = f(x_2) > y_0$. Then the points $P_1(x_1, y_1)$ and $P_2(x_2, y_2)$ lie in the strip determined by the vertical lines $x = x_0 - \varepsilon$ and $x = x_0 + \varepsilon$, and on different sides of the horizontal line $y = y_0$. If one of these points lies in the square, then we are done. Hence we may assume that both of them lie outside the square. Consider the midpoint $x_3 = \frac{x_1+x_2}{2}$ of the interval

$[x_1, x_2]$. Since the function f is additive we have that

$$y_3 = f(x_3) = f\left(\frac{x_1 + x_2}{2}\right) = \frac{f(x_1) + f(x_2)}{2} = \frac{y_1 + y_2}{2}.$$

Hence y_3 is the midpoint of the interval $[y_1, y_2]$ and therefore

$$|y_3 - y_0| \leq \frac{y_2 - y_1}{2}.$$

If the point $P_3(x_3, y_3)$ lies inside the square the theorem is proved. So, we may assume that $P_3(x_3, y_3)$ lies outside the square. If $f(x_3) > y_0$ ($f(x_3) < y_0$), we set

$$x_4 = \frac{x_1 + x_3}{2} \quad \left(x_4 = \frac{x_2 + x_3}{2}\right).$$

In both cases we have that

$$|y_4 - y_0| \leq \frac{y_2 - y_1}{2^2},$$

where $y_4 = f(x_4)$. Continuing in the same way we construct a sequence of points $P_n(x_n, y_n)$ on the graph of f such that $x_n \in (x_0 - \varepsilon, x_0 + \varepsilon)$ and

$$|y_n - y_0| \leq \frac{y_2 - y_1}{2^n}$$

for any $n \in \mathbb{N}$. It follows that $\lim\limits_{n\to\infty} y_n = y_0$ and therefore there exists $n \in \mathbb{N}$ such that $y_n \in (y_0 - \varepsilon, y_0 + \varepsilon)$. Hence the point $P_n(x_n, y_n)$ lies inside the square and the theorem is proved.

Second Proof. We shall use the fact that for any real number there is a rational number which is arbitrarily close to it. If the function $f(x)$ does not have the form $f(x) = ax$ then there exist x_1, $x_2 \neq 0$ such that

$$\frac{f(x_1)}{x_1} \neq \frac{f(x_2)}{x_2}.$$

Hence the vectors $\mathrm{v}_1 = (x_1, f(x_1))$ and $\mathrm{v}_2 = (x_2, f(x_2))$ are linearly independent and for any vector v there exist real numbers ρ_1 and ρ_2 such that

$v = \rho_1 \mathrm{v}_1 + \rho_2 \mathrm{v}_2$. Therefore there exists a vector of the form $r_1\mathrm{v}_1 + r_2\mathrm{v}_2$ with rational r_1 and r_2 which is arbitrary close to v. Since

$$r_1\mathrm{v}_1 + r_2\mathrm{v}_2 = r_1(x_1, f(x_1)) + r_2(x_2, f(x_2)) = (r_1x_1 + r_2x_2, f(r_1x_1 + r_2x_2))$$

it follows that for any point inside the square there is a point on the graph of f which is arbitrary close to it. Hence this point also lies inside the square. □

Using standard mathematical terminology the above theorem says that if the graph of an additive function is not a line then it is a dense subset of the plane which is unbounded around each of its points. The existence of such additive functions was proved by Hamel [11] in 1905 by using the so-called *Axiom of Choice* which says that for any family $\{F_\alpha\}$ of nonempty sets F_α there exists a set $F = \{x_\alpha\}$, where $x_\alpha \in F_\alpha$ for any α.
Note that the Axiom of Choice has some quite unexpected consequences that contradict normal thinking. That is why many mathematicians do not accept it and this led to the development of branches of mathematics where the Axiom of Choice is not valid.
Next we shall use the following theorem of Hamel [11] proved by means of the Axiom of Choice.

Theorem 1.3. *There exists a set H of real numbers such that:*
(a) $1 \in H$*;*
(b) if $h_1, h_2, \dots, h_n \in H$ *and* $r_1, r_2, \dots, r_n$ *are rational numbers with*

$$r_1h_1 + r_2h_2 + \cdots r_nh_n = 0,$$

then $r_1 = \cdots = r_n = 0$*;*
(*c*) *for any real number x there exist rational numbers $r_1, \dots, r_n$ and real numbers $h_1, h_2, \dots, h_n \in H$ such that $x = r_1h_1 + \cdots + r_nh_n$.*

Any set H of real numbers having the above three properties is called a *Hamel basis*. Note that for any $x \in \mathbb{R}$, $x \neq 0$, the sum in (c) is finite but the number of the summands depends on x. Hence any $x \in \mathbb{R}$ can be written in the form

$$x = \sum r_\alpha h_\alpha, \tag{1.3}$$

where $\{h_\alpha\} = H$ and only finite of the rational numbers r_α are nonzero. Suppose that $x = \sum r_\alpha h_\alpha$ and $x = \sum s_\alpha h_\alpha$. Then

$$0 = x - x = \sum (r_\alpha - s_\alpha) h_\alpha$$

and it follows from (b) that $r_\alpha = s_\alpha$. Hence the rational numbers r_α in the representation (1.3) are uniquely determined by x and can be considered as coordinates of x with respect to the Hamel basis H.
The importance of the Hamel bases for description of additive functions is manifested by the following

Theorem 1.4. *Let H be a Hamel basis and let $s : H \to \mathbb{R}$ be an arbitrary function. Then there exists a unique additive function $f : \mathbb{R} \to \mathbb{R}$ such that $f(h) = s(h)$ for any $h \in H$.*

Proof. To use the same notation is above, we set $H = \{h_\alpha\}$. Given a function $s : H \to \mathbb{R}$ we can define a function $f : \mathbb{R} \to \mathbb{R}$ in the following way. Let x be an arbitrary real number. Then $x = \sum r_\alpha h_\alpha$, where $r_\alpha \in \mathbb{Q}$ and only finite of the numbers r_α are nonzero. Set $f(x) = \sum r_\alpha s(h_\alpha)$. Then it is clear that $f(h_\alpha) = s(h_\alpha)$ for any $h_\alpha \in H$ and we have to prove that f is an additive function. Let $x = \sum r'_\alpha h_\alpha$ and $y = \sum r''_\alpha h_\alpha$, where $r'_\alpha, r''_\alpha \in \mathbb{Q}$. Then it follows from Proposition 1.1 that

$$\begin{aligned} f(x+y) &= f\left(\sum \left(r'_\alpha + r''_\alpha\right) h_\alpha\right) = \sum \left(r'_\alpha + r''_\alpha\right) f(h_\alpha) \\ &= \sum (r'_\alpha + r''_\alpha) s(h_\alpha) = \sum r'_\alpha s(h_\alpha) + \sum r''_\alpha s(h_\alpha) \\ &= f(x) + f(y) \end{aligned}$$

and the theorem is proved. □

The above theorem shows that there is an one-to-one correspondence between the set of all additive functions $f : \mathbb{R} \to \mathbb{R}$ and the set of all functions $s : H \to \mathbb{R}$. We should note that the additive functions not of the form $f(x) = ax$, $a \in \mathbb{R}$, correspond to the functions $s : H \to \mathbb{R}$ such that the function $\frac{s(h)}{h}$ is not constant (if s corresponds to the function $f(x) = ax$ then $s(h) = ah$ for any $h \in H$ and $\frac{s(h)}{h} = a$). Hence any such function determines an additive function which does not satisfy the conditions (i)–(iv) of Theorem 1.1.

Example 1.3. Let $H = \{h_\alpha\}$ be a Hamel basis. Consider the additive function $f : \mathbb{R} \to \mathbb{R}$ determined by the function $s : H \to \mathbb{R}$ defined by:

$$s(h_\alpha) = \begin{cases} h_\alpha & \text{if } h_\alpha \neq 1 \\ 2 & \text{if } h_\alpha = 1. \end{cases}$$

Then according to Proposition 1.1 we have that $f(x) = xf(1) = xs(1) = 2x$ for any $x \in \mathbb{Q}$. Note that the function $f : \mathbb{R} \to \mathbb{R}$ is not of the form $f(x) = ax$ since $\frac{s(1)}{1} = 2$ but $\frac{s(h_\alpha)}{h_\alpha} = 1$ for $h_\alpha \neq 1$.

One can show (see eg. [19]) that for any Hamel basis H there exists a bijection $\varphi : [0,1] \to H$. If we set $\varphi(t) = h_t$, then $H = \{h_t | t \in [0,1]\}$. This observation leads to constructions of additive functions having interesting properties.

Example 1.4. Let $f : \mathbb{R} \to \mathbb{R}$ be the additive function determined by the function $s : H \to \mathbb{R}$ defined by:

$$s(h_t) = \begin{cases} h_t & \text{if } t \in \left[0, \frac{1}{2}\right] \\ -h_t & \text{if } t \in \left(\frac{1}{2}, 1\right]. \end{cases}$$

Then it is easy to check that $f(f(x)) = x$ for any $x \in \mathbb{R}$.
Another interesting example of a discontinuous additive function is obtained by considering the function $s : H \to \mathbb{R}$ defined by

$$s(h_t) = \begin{cases} h_t & \text{if } t \in \left[0, \frac{1}{2}\right] \\ 0 & \text{if } t \in \left(\frac{1}{2}, 1\right]. \end{cases}$$

Then $f(f(x)) = f(x)$ for all $x \in \mathbb{R}$.

Finally we shall show that any additive function on $[0, \infty)$ can be extended in a unique way to an additive function on $\mathbb{R}$.

Theorem 1.5. *Let $g : [0, \infty) \to \mathbb{R}$ be an additive function. Then there exists a unique additive function $f : \mathbb{R} \to \mathbb{R}$ such that $f(x) = g(x)$ for all $x \in [0, \infty)$.*

Proof. Let $f : \mathbb{R} \to \mathbb{R}$ be an additive function such that $f(x) = g(x)$ for all $x \in [0, \infty)$. Since $f(-x) = -f(x)$ for all $x \in \mathbb{R}$ we see that

$$f(x) = \begin{cases} g(x) & \text{if } x \geq 0 \\ -g(-x) & \text{if } x \leq 0 \end{cases} \tag{1.4}$$

This proves the uniqueness of the extension f of g. To prove the existence let $f : \mathbb{R} \to \mathbb{R}$ be the function defined by (1.4). We shall prove that f is an additive function. If $x \geq 0$ and $y \geq 0$, then $x + y \geq 0$ and we get

$$f(x) + f(y) = g(x) + g(y) = g(x + y) = f(x + y).$$

If $x \leq 0$ and $y \leq 0$, then

$$f(x) + f(y) = -(g(-x) + g(-y)) = -g(-x - y) = f(x + y).$$

Let now $x \leq 0$ and $y \geq 0$. If $x + y \geq 0$ then

$$f(y) = g(y) = g((x + y) + (-x)) = g(x + y) + g(-x) = f(x + y) - f(x)$$

and therefore $f(x+y) = f(x)+f(y)$. Proceeding in the same way we see that the latter identity also holds true if $x+y \leq 0$. Hence $f : \mathbb{R} \to \mathbb{R}$ is an additive function and the theorem is proved. □

In particular, the above theorem shows that there is a one-to-one correspondence between the class of additive functions on $[0, \infty)$ and the class of additive functions on $\mathbb{R}$.

The additive Cauchy equation (1.1) is related to the additive structure of the set of real numbers and as we saw in Theorem 1.1 it can be used to characterize the class of linear functions $f(x) = ax$, $a \in \mathbb{R}$.

Next we consider some analogous functional equations that are related to both the additive and the multiplicative structure of $\mathbb{R}$. As we shall see these equations can be used to characterize the logarithmic function $\lg x$, the exponential function a^x, $a > 0$, and the power function x^a, $a \in \mathbb{R}$.

1.2 Logarithmic Cauchy Equation

A well known property of the logarithmic function $\lg x$ is that

$$\lg xy = \lg x + \lg y \text{ for all } x, y \in (0, \infty).$$

This identity suggests considering the functional equation

$$f(xy) = f(x) + f(y), \quad x, y \in (0, \infty), \tag{1.5}$$

which is called the logarithmic Cauchy equation [2]. We shall show that its solutions are related to the solutions of the additive Cauchy equation (1.1) by means of the logarithmic function.

Theorem 1.6. *Any solution $f : (0, \infty) \to \mathbb{R}$ of the logarithmic Cauchy equation has the form $f(x) = g(\lg x)$, where $g : \mathbb{R} \to \mathbb{R}$ is an additive function.*

Proof. We shall use the fact that for any $x \in (0, \infty)$ there is a unique $u \in \mathbb{R}$ such that $x = 10^u$. In fact $u = \lg x$. For any $u \in \mathbb{R}$ set $g(u) = f(10^u)$. Then

$$g(u + v) = f(10^{u+v}) = f(10^u \cdot 10^v) = f(10^u) + f(10^v) = g(u) + g(v).$$

Hence $g : \mathbb{R} \to \mathbb{R}$ is an additive function such that

$$f(x) = g(\lg x), \; x \in (0, \infty). \qquad \square$$

An immediate consequence of Theorems 1.1 and 1.6 is the following characterization of the logarithmic function:

Corollary 1.2. *Let $f : (0, \infty) \to \mathbb{R}$ be a solution of the logarithmic Cauchy equation having one of the following properties:*
(i) f is bounded above (below) on an interval;
(ii) f is increasing (decreasing) on an interval;
(iii) f is continuous at a point.
Then $f(x) = a \lg x$, where $a \in \mathbb{R}$.

We next consider the logarithmic Cauchy equation for functions $f : D \to \mathbb{R}$ whose domain D is different from $(0, \infty)$. Note first that if $D = [0, \infty)$ then $f(x) = 0$ for all $x \in [0, \infty)$. Indeed, setting $y = 0$ in (1.5) we get

$$f(0) = f(x) + f(0), \text{ i.e. } f(x) = 0.$$

The case $D = \mathbb{R}\backslash\{0\}$ is more interesting.

Theorem 1.7. *Any solution $f : \mathbb{R}\backslash\{0\} \to \mathbb{R}\backslash\{0\}$ of the logarithmic Cauchy equation has the form $f(x) = g(\lg|x|)$, where $g : \mathbb{R} \to \mathbb{R}$ is an additive function.*
If f has one of the properties (i)–(iii) in Corollary 1.2 then $f(x) = a \lg|x|$, where $a \in \mathbb{R}$.

Proof. Setting $x = y = 1$ in (1.5) we get that $f(1) = 0$. Then setting $x = y = -1$ gives $f(-1) = 0$. Moreover, it follows from Theorem 1.6 that $f(x) = g(\lg x)$ for any $x \in \mathbb{R}\backslash\{0\}$, where $g : \mathbb{R} \to \mathbb{R}$ is an additive function. Now for any $x \in \mathbb{R}\backslash\{0\}$ we have that

$$f(x) = f(|x| \cdot \operatorname{sgn} x) = f(|x|) + f(\operatorname{sgn} x) = f(|x|) = g(\lg|x|). \qquad \square$$

Another interesting case is $D = (0, 1]$.

Theorem 1.8. *Any solution $f : (0, 1] \to \mathbb{R}$ of the logarithmic Cauchy equation has the form $f(x) = g(-\lg x)$, where $g : [0, \infty) \to \mathbb{R}$ is an additive function. If g satisfies one of the conditions (i)–(iii) of Corollary 1.2, then $g(x) = a \lg x$, where $a \in \mathbb{R}$.*

Proof. For any $x \in (0, 1]$ there is a unique $u \in [0, \infty)$ such that $x = 10^{-u}$. Then setting $g(u) = f(10^{-u})$ for any $u \in [0, \infty)$ we see as in the proof of Theorem 1.6 that $f : [0, \infty) \to \mathbb{R}$ is an additive function such that

$$f(x) = g(-\lg x) \text{ for any } x \in (0, 1].$$

The last statement follows from Theorem 1.1. $\square$