Preface

As a sequel to *113 Geometric Inequalities from the AwesomeMath Summer Program*, this book extends the themes discussed in the former book and broadens a problem-solver's competitive arsenal. The beauty of geometry is explored from another perspective, namely the mechanics associated with defining and breaking down the aspects of such visual and numerically-involved problems. One may wonder how complex figures can be described through linear transformations; nevertheless, many creative and powerful techniques are featured. Strategies from multiple fields, such as Algebra, Calculus, and pure Geometry provide the reader with a varied subset of methods useful in mathematics competitions. Starting with the fundamentals such as the triangle inequality and "broken lines", the book progresses increasingly to more sophisticated machinery such as the Averaging Method, Quadratic Forms, Finite Fourier Transforms, Level Curves, the Erdös-Mordell and Brunn-Minkowski Inequalities, and the Isoperimetric Theorem, to name a few. Despite the focus on contest-style problems, rich theory and generalizations accompany the aforementioned topics to supply the curious reader with a rigorous exploration of fields associated with geometric inequalities. To help the student better assimilate these many techniques, intuitive motivations and well-organized solutions are listed in relative order of difficulty. Frequently, even three solutions are provided to link the many interconnected areas of mathematics that converge in unexpected geometric themes. The study of geometric inequalities will also indirectly strengthen the reader's ability to analyze, dissect, and invent creative methods, all skills that are necessary to succeed in mathematics competitions. We hope that this book will illuminate some underappreciated geometric arguments and fortify the reader's mathematical toolkit.

Many thanks to Mircea Becheanu, Gabriel Dospinescu, and Christian Yankov for several useful discussions on some of the topics featured in the book. And thank you to Chris Jeuell and Adrian Andreescu for significantly improving and polishing the text.

Enjoy the book!

Titu Andreescu, Oleg Mushkarov

Contents

1 The Triangle Inequality **1**
1.1 Broken Lines 1
1.2 Shortest Paths 21
1.3 The Averaging Method 30
1.4 Practice Problems 39

2 Algebraic Methods **45**
2.1 Algebraic Inequalities 45
2.2 The Dot Product 55
2.3 Complex Numbers 66
2.4 Finite Fourier Transforms 73
2.5 Quadratic Forms 85
2.6 Practice Problems 100

3 Analytic Methods **105**
3.1 Employing Calculus 105
3.2 Partial Variations 114
3.3 Level Curves 124
3.4 Distances Between Sets 140
3.5 Practice Problems 155

4 The Fundamental Inequality in Triangles **161**
4.1 An Analytic Proof of The Fundamental Inequality 162
4.2 An Algebraic Proof of The Fundamental Inequality 165

4.3 A Geometric Proof of The Fundamental Inequality 168
4.4 Sharp Inequalities in Triangles 169
4.5 Proving Algebraic Inequalities 176
4.6 Practice Problems 179

5 The Erdös-Mordell Inequality 183
5.1 Seven Proofs of the Erdös-Mordell Inequality 183
5.2 Examples of Erdös-Mordell Type Inequalities 190
5.3 A Generalization of the Erdös-Mordell Inequality 201
5.4 The Erdös-Mordell Inequality for Polygons 205
5.5 Practice Problems 208

6 Area Inequalities 211
6.1 Inscribed and Circumscribed Polygons 211
6.2 Malfatti's Marble Problem 225
6.3 The Brunn-Minkowski Inequality 238
6.4 Practice Problems 250

7 Isoperimetric Problems 253
7.1 The Isoperimetric Theorem 253
7.2 Isoperimetric Problems 258
7.3 Practice Problems 263

8 Hints and Solutions 265
8.1 The Triangle Inequality 265
8.2 Algebraic Methods 287
8.3 Analytic Methods 307
8.4 The Fundamental Inequality in Triangles 346
8.5 The Erdös-Mordell Inequality 371
8.6 Area Inequalities 387
8.7 Isoperimetric Problems 405

Bibliography 419

Chapter 1

The Triangle Inequality

The triangle inequality and its generalization for broken lines is one of the most basic tools for proving distance inequalities in the plane and space. In this chapter we consider several classical examples of such inequalities as well as some related practical problems for shortest paths in the plane. The last section is devoted to the so-called averaging method for proving inequalities for the lengths of broken lines.

1.1 Broken Lines

The triangle inequality says that for any three points A, B, C, we have

$$AB + BC \geq CA.$$

It follows by induction that for any points $A_1, A_2, \ldots, A_n$, $n \geq 3$ in the plane (space) (Fig. 1.1) the following generalized triangle inequality holds true:

$$A_1A_2 + A_2A_3 + \cdots + A_{n-1}A_n \geq A_1A_n. \tag{1.1}$$

The equality in (1.1) occurs if and only if points $A_2, \ldots, A_{n-1}$ lie on the segment A_1A_n in this order.

We will also use the vector analog of the triangle inequality which says that for any vectors $\overrightarrow{a_1}, \overrightarrow{a_2}, \ldots, \overrightarrow{a_n}$ in the plane (space), we have:

$$|\overrightarrow{a_1}| + |\overrightarrow{a_2}| + \cdots + |\overrightarrow{a_n}| \geq |\overrightarrow{a_1} + \overrightarrow{a_2} + \cdots + \overrightarrow{a_n}|.$$

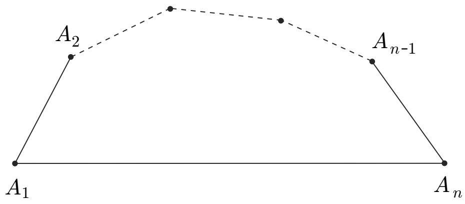

Figure 1.1

Equality occurs if and only if the vectors $\overrightarrow{a_1}, \overrightarrow{a_2}, \ldots, \overrightarrow{a_n}$ are collinear and equally oriented.

Example 1.1. Let M be a point inside triangle ABC. Prove that:

(a) $MA + MB < CA + CB$;

(b) $MA + MB + MC < \max(AB + BC, BC + CA, CA + AB)$.

Solution. (a) Let N be intersection of lines AM and BC (Fig. 1.2). Then by the triangle inequality, $BM < MN + NB$ and $AN < CA + CN$. Hence

$$AM + BM < AM + MN + BN = AN + BN < CA + CN + NB = CA + CB.$$

(b) Without loss of generality, let $AB \leq BC \leq CA$. Draw the lines through M parallel to the sides of the triangle and denote by A_1 and A_2, B_1 and B_2, C_1 and C_2 the points where they intersect BC, CA, AB, respectively (Fig. 1.3). Then triangles A_1A_2M, MB_1B_2, C_2MC_1 are similar, and their shortest sides are MA_1, MB_2, C_1C_2, respectively. This together with the triangle inequality implies

$$\begin{aligned} MA + MB + MC &< (AB_2 + B_2M) + (MA_1 + A_1B) + (MA_2 + A_2C) \\ &< (AB_2 + B_2B_1) + (A_1A_2 + A_1B) + (CB_1 + A_2C) \\ &= AC + BC. \end{aligned}$$

Here we have used the fact that $MA_2 = B_1C$. (Why?) $\square$

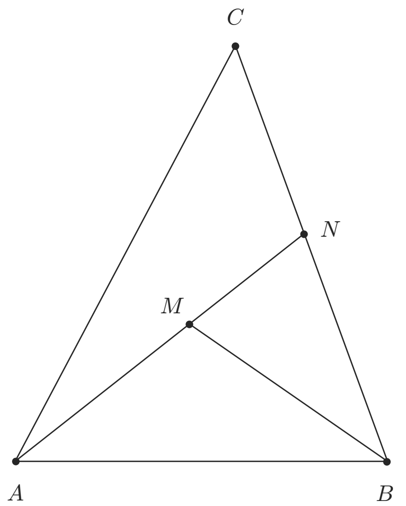

Figure 1.2

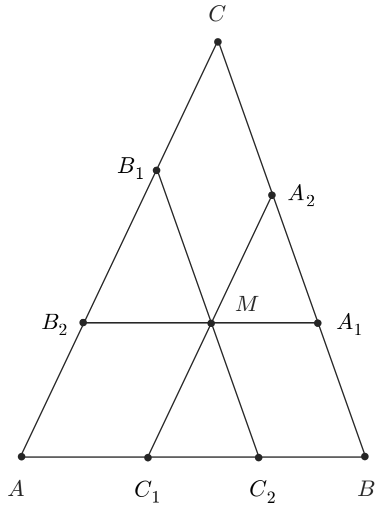

Figure 1.3

Example 1.2. Let M be a point on a segment AB and K a point in the plane (space). Prove that:

(a) If M is the midpoint of AB, then

$$KM \leq \frac{KA + KB}{2}.$$

(b) If $\frac{MB}{AB} = \lambda$, $0 < \lambda < 1$, then

$$KM \leq \lambda \cdot KA + (1 - \lambda)KB.$$

Equality in (a) and (b) is attained if and only if the points A, B, K are collinear and K lies outside the segment AB.

(c) If G is the centroid of triangle ABC, then

$$KG \leq \frac{KA + KB + KC}{3}.$$

Equality is attained if and only if the points A, B, C, K are collinear and K lies outside the segments AB, BC, and CA.

Solution. (a) Consider the point N such that $ANBK$ is a parallelogram (Fig. 1.4). Then

$$KM = \frac{1}{2}KN \leq \frac{1}{2}(KB + BN) = \frac{1}{2}(KB + KA).$$

Note also that this inequality is a special case of (b) for $\lambda = \dfrac{1}{2}$.

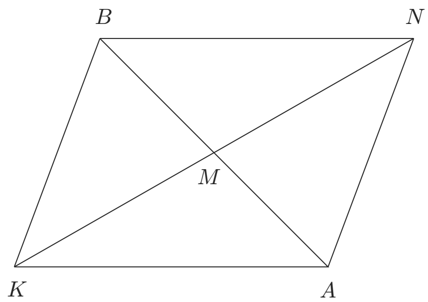

Figure 1.4

(b) Let A_1 and B_1 be points on the segments KA and KB such that $MA_1 \parallel KB$ and $MB_1 \parallel KA$ (Fig. 1.5). Then

$$MB_1 = \frac{MB}{AB} \cdot KA = \lambda \cdot KA, \;\; MA_1 = \frac{MA}{AB} \cdot KB = (1 - \lambda)KB$$

and the desired inequality follows from the triangle inequality:

$$KM \leq MB_1 + B_1K = MB_1 + MA_1 = \lambda \cdot KA + (1 - \lambda)KB.$$

This inequality also follows from the identity

$$\overrightarrow{KM} = \lambda\overrightarrow{KA} + (1 - \lambda)\overrightarrow{KB}$$

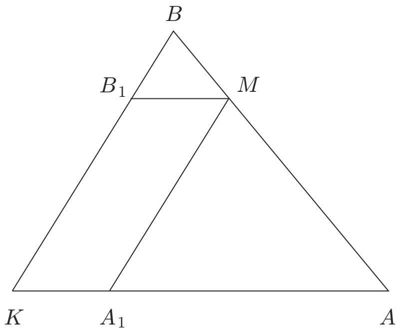

Figure 1.5

and the triangle inequality for vectors. Equality in (a) and (b) is attained if and only if the vectors $\overrightarrow{KA}$ and $\overrightarrow{KB}$ are collinear, i.e., the points A, B, K are collinear and K lies outside the segment AB. □

(c) Let M be the midpoint of segment AB. We know that $\dfrac{GM}{CG} = \dfrac{1}{3}$. Hence from (b) and (a), it follows that

$$KG \le \frac{1}{3}(KC + 2KM) \le \frac{1}{3}(KC + KA + KB).$$

Another way to prove the desired inequality is to use the identity

$$\overrightarrow{KA} + \overrightarrow{KB} + \overrightarrow{KC} = 3\overrightarrow{KG}$$

and the triangle inequality for vectors. Equality in (c) is attained if and only if the vectors $\overrightarrow{KA}, \overrightarrow{KB}, \overrightarrow{KC}$ are collinear, i.e., the points A, B, C, K are collinear and K lies outside the segments AB, BC, and CA. □

Example 1.3. (Heron's problem) Two points A and B lie on one side of a straight line l. Find a point C on l such that $CA + CB$ is minimized.

Solution. Let B' be the reflection of B in l (Fig. 1.6). Then $BG = B'G$ and the triangle inequality for $\triangle ACB'$ implies

$$AC + CB = AC + CB' \ge AB'.$$

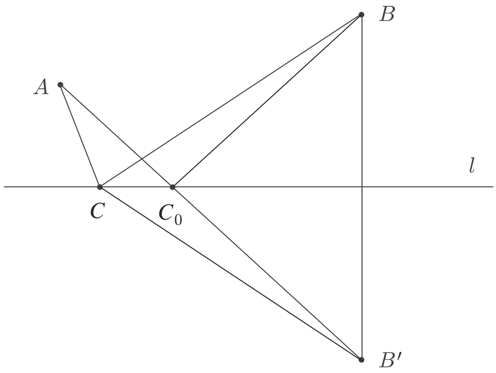

Figure 1.6

Equality occurs precisely when C is the intersection point C_0 of l and the line segment AB'. □

Remark. The above problem has been considered about 2000 years ago by Heron who stated that the shortest distance between A and B via a line l is exactly the path traversed by a ray of light emitted from A and observed at B. From here he deduced that when light is reflected in a mirror, the angle of incidence is equal to the angle of reflection.

Example 1.4. (Ptolemy's inequality) For any four points A, B, C, D in the plane, we have

$$AC \cdot BD \leq AB \cdot CD + BC \cdot AD.$$

Equality holds if and only if $ABCD$ is a cyclic quadrilateral.

First Solution. We may assume that B lies inside $\angle ADC$. On rays $\overrightarrow{DA}$, $\overrightarrow{DB}$, $\overrightarrow{DC}$, consider the points A_1, B_1, C_1, respectively, such that

$$DA_1 = \frac{1}{DA}, \quad DB_1 = \frac{1}{DB}, \quad DC_1 = \frac{1}{DC}.$$

Then $\triangle ABC \sim \triangle A_1B_1C_1$ and so

$$A_1B_1 = \frac{AB}{DA \cdot DB}, \quad B_1C_1 = \frac{BC}{DB \cdot DC}, \quad C_1A_1 = \frac{CA}{DC \cdot DA}.$$

The desired inequality follows from the triangle inequality:

$$A_1B_1 + B_1C_1 \geq A_1C_1.$$

Equality holds if and only if B_1 lies on the segment A_1C_1, that is, when

$$\angle BAD + \angle BCD = \angle A_1B_1D + \angle C_1B_1D = 180^\circ. \qquad \square$$

Second Solution. See the solution of Example 2.18 using complex numbers.
$\square$

Example 1.5. (Pompeiu's theorem) Let ABC be an equilateral triangle and let M be a point in its plane. Prove that the segments AM, BM, CM are side lengths of a triangle. Also prove that this triangle is degenerate if and only if M lies on the circumcenter of triangle ABC.

First Solution. By Ptolemy's inequality for points A, M, B, C, it follows that

$$AB \cdot CM \leq AM \cdot BC + BM \cdot AC.$$

Since $AB = BC = CA$, we get $CM \leq AM+BM$. Similarly, $BM \leq CM+AM$ and $AM \leq BM + CM$. We have equality in one of these inequalities, say in the first one, if and only if $AMBC$ is a cyclic quadrilateral. $\square$

Second Solution. Consider the rotation of 60° about A, and let M_1 be the image of M (Fig. 1.7). Then $AM = MM_1$, $CM_1 = BM$, and $\triangle MM_1C$ is the desired triangle. $\square$
Note that it degenerates if and only if points M_1, C, M are collinear which implies that M lies on the circumcircle of triangle ABC. (Why?)

Example 1.6. (IMO 1995) Let $ABCDEF$ be a convex hexagon with $AB = BC = CD$ and $DE = EF = FA$, such that $\angle BCD = \angle EFA = 60^\circ$. Suppose G and H are points in the interior of the hexagon such that $\angle AGB = \angle DHE = 120^\circ$. Prove that $AG + GB + GH + DH + HE \geq CF$.

Solution. Triangles BCD and EFA are equilateral, and hence BE is an axis of symmetry of $ABDE$. Let C', F' respectively be the points symmetric to C, F with respect to BE. Points G and H lie on the circumcircles of ABC' and

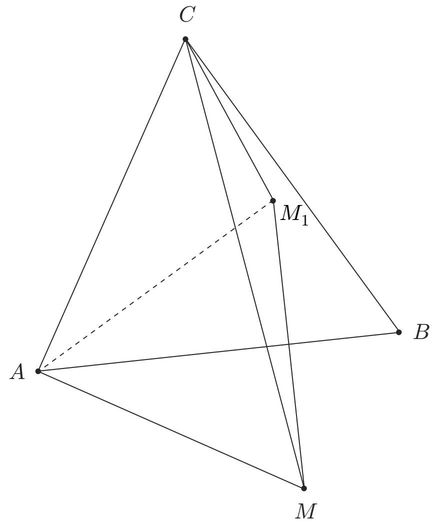

Figure 1.7

DEF', respectively, because, for instance, $\angle AGB = 120° = 180° - \angle AC'B$. Hence from Ptolemy's Theorem, we have $AG + GB = C'G$ and $DH + HE = HF'$. Therefore

$$AG + GB + GH + DH + HE = C'G + GH + HF' \geq C'F' = CF,$$

with equality if and only if both G and H lie on $C'F'$. □

Remark. By Ptolemy's inequality (Example 1.4),

$$AG + GB \geq C'G, \quad DH + HE \geq HF',$$

so the result holds without the condition $\angle AGB = \angle DHE = 120°$.

Example 1.7. Among all quadrilaterals $ABCD$ with $AB = 3, CD = 2$, and $\angle AMB = 120°$, where M is the midpoint of CD, find the one of minimal perimeter.

Solution. Let C' and D' be the reflections of C and D in the lines BM and AM, respectively (Fig. 1.8). Then triangle $C'MD'$ is equilateral because

$$C'M = D'M = \frac{1}{2}CD \quad \text{and} \quad \angle C'MD' = 180^\circ - 2\angle CMB - 2\angle DMA = 60^\circ.$$

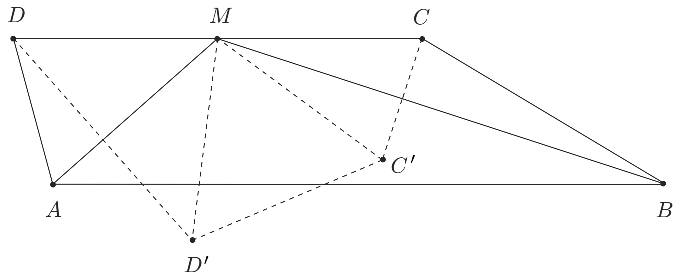

Figure 1.8

Hence

$$AD + \frac{1}{2}CD + CB = AD' + D'C' + C'B \geq AB.$$

It follows that $AD + CB \geq AB - \frac{1}{2}CD = 2$. Thus $AB + BC + CD + DA \geq 7$, with equality if and only if C' and D' lie on AB.
In the latter case, $\angle ADM = \angle AD'M = 120^\circ$, $\angle BCM = \angle BC'M = 120^\circ$, and $\angle AMD = 60^\circ - \angle CMB = \angle CBM$. Hence triangles AMD and MBC are similar, implying

$$AD \cdot BC = \left(\frac{CD}{2}\right)^2 = 1.$$

On the other hand, $AD + BC = 2$, and we conclude that $AD = BC = 1$. Therefore the quadrilateral $ABCD$ of minimum perimeter is an isosceles trapezoid with sides $AB = 3, BC = AD = 1$, and $CD = 2$ (Fig. 1.9). □

The next problem was first stated by Giovanni Fagnano in 1775.

Example 1.8. (Fagnano's problem) Prove that of all triangles inscribed in a given acute triangle, the orthic triangle has the least perimeter.

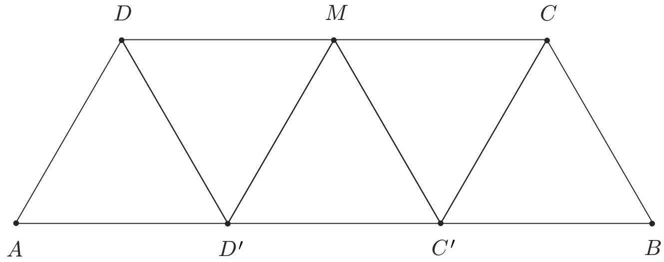

Figure 1.9

First Solution. Let ABC be the given triangle and let M, N, P be arbitrary points on the sides AB, BC, CA, respectively. Denote by E and F the respective feet of the perpendiculars from M to AC and BC (Fig. 1.10).

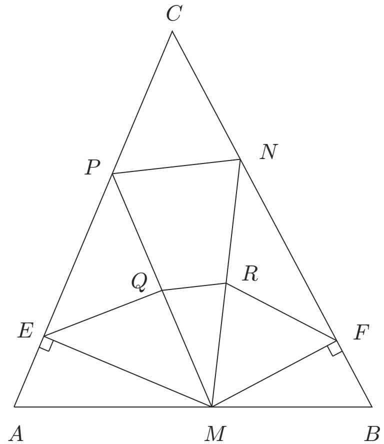

Figure 1.10

Then the quadrilateral $MFCE$ is inscribed in the circle with diameter CM and therefore $EF = CM \sin\angle C$. Let Q and R be the midpoints of MP and MN, respectively. Then

$$MN + NP + PM = 2FR + 2QR + 2QE \geq 2EF = 2CM \sin\angle C.$$

Let AA_1, BB_1, CC_1 be the altitudes of triangle ABC, and let E_1 and F_1 be the feet of the perpendiculars from C_1 to AC and BC, respectively (Fig. 1.11). Then $E_1F_1 = CC_1 \sin\angle C$. Denote by Q_1 and R_1 the midpoints of C_1B_1 and C_1A_1, respectively.

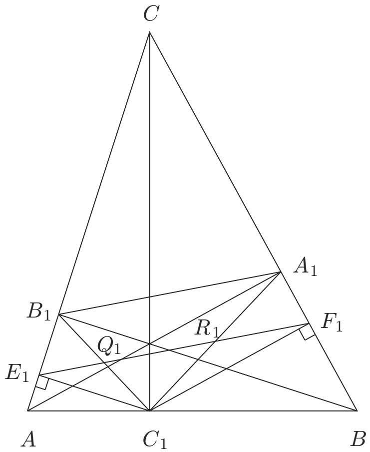

Figure 1.11

Then

$$\angle E_1Q_1B_1 = 2\angle E_1C_1B_1 = 2\angle C_1B_1B = \angle C_1B_1A_1,$$

which shows that $E_1Q_1 \parallel A_1B_1$. Similarly, $F_1R_1 \parallel A_1B_1$. It follows that E_1, Q_1, R_1, F_1 are collinear and

$$A_1B_1 + B_1C_1 + C_1A_1 = 2Q_1R_1 + 2Q_1E_1 + 2R_1F_1 = 2E_1F_1 = 2CC_1 \sin\angle C.$$

Thus

$$MN + NP + PM = 2CM \sin\angle C \geq 2CC_1 \sin\angle C = A_1B_1 + B_1C_1 + C_1A_1.$$

Hence of all triangles MNP inscribed in triangle ABC, the orthic triangle $A_1B_1C_1$ has the least perimeter. □

Remark. Fagnano's problem can also be solved in the case when the given triangle is not acute-angled. Assume, for example, that $\angle ACB \geq 90°$. It is not difficult to see that in this case, triangle MNP with minimal perimeter occurs when $N = P = C$ and M is the foot of the altitude of triangle ABC through C. In this case, triangle MNP is degenerate.

We now consider the analog of Fagnano's problem for convex polygons. Note that for any $n \geq 4$, there are convex n-gons that have no inscribed n-gons of minimal perimeter.

Example 1.9. Let $\mathcal{A}$ be a convex n-gon with vertices $A_1, A_2, \ldots, A_n$ and let $\mathcal{B}$ be an inscribed n-gon with vertices $B_i \in A_iA_{i+1}, 1 \leq i \leq n, A_{n+1} = A_1$. Then $\mathcal{B}$ has minimal perimeter amongst all inscribed n-gons in $\mathcal{A}$ if and only if $\angle B_nB_1A_1 = B_2B_1A_2,\ \angle B_1B_2A_2 = B_3B_2A_3, \ldots, \angle B_{n-1}B_nA_n = \angle B_1B_nA_1$.

Solution. Assume that $\mathcal{B}$ has minimal perimeter amongst all inscribed n-gons in $\mathcal{A}$ but the given condition is not satisfied. Let, for example, $\angle B_nB_1A_1 \neq \angle B_2B_1A_2$. Consider point B_2' symmetric to B_2 with respect to the line A_1A_2 (Fig. 1.12). Denote by B the intersection of lines $B_2'B_n$ and A_1A_2. Since $\angle B_2BM = \angle B_2'BM = \angle B_nBA_1 \neq \angle B_2B_1A_2$ points B and B_1 differ.

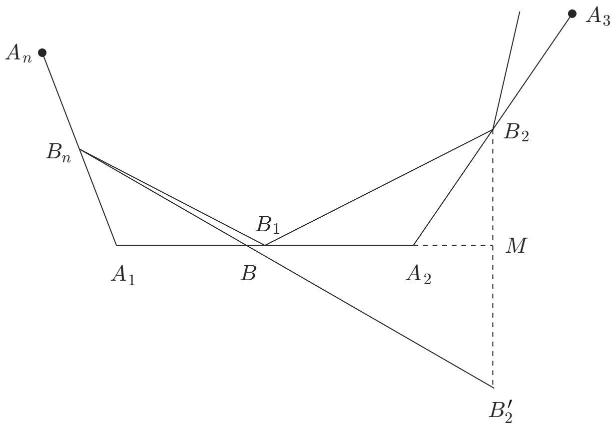

Figure 1.12

Take a point B_1' on the segment BB_1 so that B_1' lies on the side A_1A_2. Consider points $B_1', B_2, \ldots, B_n$.

According to Example 1.1, (i), $B_2'B_1' + B_1'B_n < B_2'B_1 + B_1B_n$,
thus $B_2B_1' + B_1'B_n < B_2B_1 + B_1B_n$, which means that

$$B_1'B_2 + B_2B_3 + \cdots + B_{n-1}B_n + B_nB_1' < B_1B_2 + B_2B_3 + \cdots + B_{n-1}B_n + B_nB_1,$$

a contradiction.
Conversely, let

$$\angle B_nB_1A_1 = B_2B_1A_2 = \beta_1,\ \angle B_1B_2A_2 = B_3B_2A_3 = \beta_2, \ldots,$$

$$\angle B_{n-1}B_nA_n = \angle B_1B_nA_1 = \beta_n.$$

Consider an arbitrary inscribed n-gon $C_1C_2\ldots C_n$ in $\mathcal{A}$. Draw through vertices $A_1, A_2, \ldots, A_n$ lines $l_1, l_2, \ldots, l_n$ parallel to sides $B_nB_1, B_1B_2, \ldots, B_{n-1}B_n$, respectively. Let C_1' and C_1'' be the orthogonal projections of C_1 on lines l_1 and let l_2, C_2' and C_2'' be the orthogonal projections of C_2 on lines l_2 and l_3, etc., and let C_n' and C_n'' be the orthogonal projections of C_n on lines l_n and l_1. We have

$$C_1C_2 + C_2C_3 + \cdots + C_{n-1}C_n + C_nC_1 \geq C_1''C_2' + C_2''C_2' + \cdots + C_{n-1}''C_n' + C_n''C_1'$$

$$= (A_2C_1\cos\beta_1 + A_2C_2\cos\beta_2) + \cdots + (C_1A_1\cos\beta_1 + C_nA_1\cos\beta_n)$$

$$= A_1A_2\cos\beta_1 + A_2A_3\cos\beta_2 + \cdots + A_nA_1\cos\beta_n$$

$$= (A_2B_1\cos\beta_1 + B_1A_1\cos\beta_1) + \cdots + (A_1B_n\cos\beta_n + A_nB_n\cos\beta_n)$$

$$= B_1B_2 + B_2B_3 + \cdots + B_{n-1}B_n + B_nB_1.$$

Hence

$$C_1C_2 + C_2C_3 + \cdots + C_{n-1}C_n + C_nC_1 \geq B_1B_2 + B_2B_3 + \cdots + B_{n-1}B_n + B_nB_1.$$

The equality is attained if and only if

$$C_1C_2 \parallel B_1B_2,\ C_2C_3 \parallel B_2B_3,\ \ldots,\ C_{n-1}C_n \parallel B_{n-1}B_n,\ C_nC_1 \parallel B_nB_1.$$

For odd n, the equality holds if $C_1 = B_1, C_2 = B_2, \ldots, C_n = B_n$. $\square$

The next problem has been first raised by Fermat in a private letter to Torricelli, who solved it.

Example 1.10. (Fermat's problem) Given points A, B, C in the plane, find all points X such that the sum of distances from X to A, B, C is minimized.

First Solution. For every point X in the plane, we set

$$t(X) = XA + XB + XC.$$

It is easy to see that if X is outside triangle ABC, then there is a point X' such that $t(X') < t(X)$. Indeed, in this case, one of the lines AB, BC, CA, say AB, is such that triangle ABC and X lie on different sides of this line (Fig. 1.13).

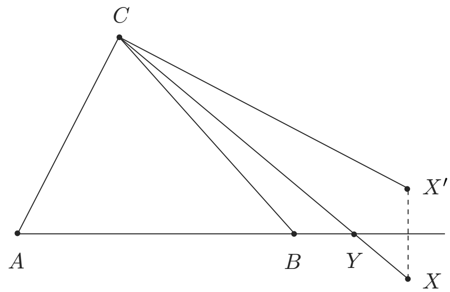

Figure 1.13

Consider the reflection X' of X in AB. We have $AX' = AX$, $BX' = BX$. Also, segment CX intersects line AB at some point Y, and $XY = X'Y$. Now the triangle inequality gives

$$CX' < CY + X'Y = CY + XY = CX,$$

implying $t(X') < t(X)$.
So we may restrict our attention to points X in the interior or on the boundary of triangle ABC. Without loss of generality, we assume that $\angle C \geq \angle A \geq \angle B$. Then $\angle A$ and $\angle B$ are both acute angles. Denote by φ the 60° rotation counterclockwise about A. For a point M in the plane, let $M' = \varphi(M)$. Then triangle AMM' is equilateral. In particular, triangle ACC' is equilateral.

Consider an arbitrary point X in triangle ABC. Then $AX = XX'$, while $\varphi(X) = X'$ and $\varphi(C) = C'$ imply $CX = C'X'$. Consequently,

$$t(X) = BX + XX' + X'C',$$

that is, $t(X)$ equals the length of the broken line $BXX'C'$.

We now consider three cases.

Case 1. $\angle C < 120°$. Then $\angle BCC' = \angle C + 60° < 180°$. Since $\angle A < 90°$, we have $\angle BAC' < 180°$, so the segment BC' intersects side AC at some point D (Fig. 1.14).

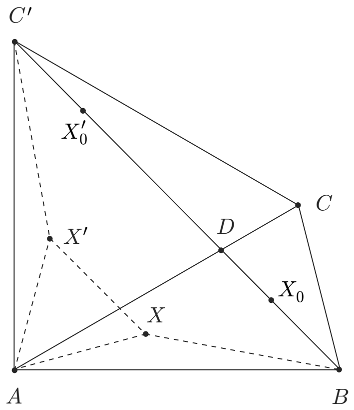

Figure 1.14

Denote by X_0 the intersection of BC' with the circumcircle of triangle ACC'. Then X_0 lies in the interior of the line segment BD and X_0' lies on $C'X_0$ since $\angle AX_0C' = \angle ACC' = 60°$. Moreover,

$$t(X_0) = BX_0 + X_0X_0' + X_0'C' = BC',$$

so $t(X_0) \leq t(X)$ for every point X in triangle ABC. Equality occurs only if both X and X' lie on BC', which is possible only when $X = X_0$. Notice that the point X_0 constructed above satisfies

$$\angle AX_0C = \angle AX_0B = \angle BX_0C = 120°.$$

It is called *Fermat's first point* or the *Fermat-Torricelli point* of triangle ABC.

Case 2. $\angle C = 120°$. In this case, the line segment BC' contains C and

$$t(X) = BX + XX' + X'C' = BC'$$

precisely when $X = C$.

Remark. Cases 1 and 2 also follow by Pompeiu's theorem (Example 1.5). Indeed, triangle ACC' is equilateral and

$$t(X) = AX + BX + CX \geq C'X + BX \geq C'B.$$

Case 3. $\angle C > 120°$. Then BC' has no common points with the side AC (Fig. 1.15).

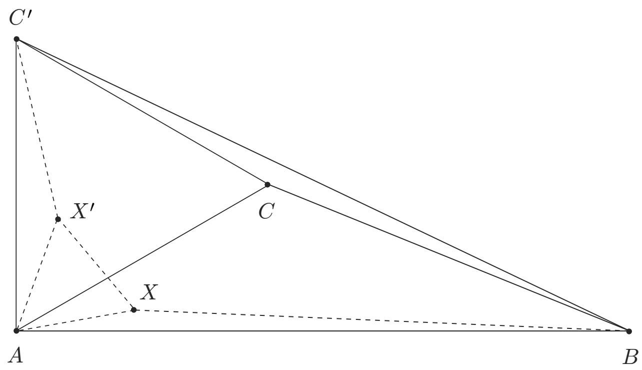

Figure 1.15

If $AX \geq AC$, then the triangle inequality gives

$$t(X) = AX + BX + CX \geq AC + BC.$$

If $AX < AC$, then X' lies in triangle ACC' and

$$t(X) = BX + XX' + X'C' \geq BC + CC' = BC + AC$$

since C lies in the quadrilateral $BC'X'X$ (Fig. 1.11). In both cases, equality occurs precisely when $X = C$.

In conclusion, if all angles of triangle ABC are less than $120°$, then $t(X)$ is minimal when X coincides with Fermat-Torricelli point of triangle ABC. If one of the angles of triangle ABC is not less than $120°$, then $t(X)$ is minimized when X coincides with the vertex of that angle. □

Second Solution. The following elegant solution of Fermat's problem is attributed to Torricelli. It is based on the well-known fact that the sum of distances from an interior point of an equilateral triangle to its sides is equal to the altitude of the triangle.

Assume that all angles of triangle ABC are less than $120°$ and denote by P its Fermat-Torricelli point. The perpendiculars to AP, BP, CP through A, B, C, respectively, determine an equilateral triangle DEF (Fig. 1.16) since, for example, $\angle FDE = 180° - \angle APB = 60°$.

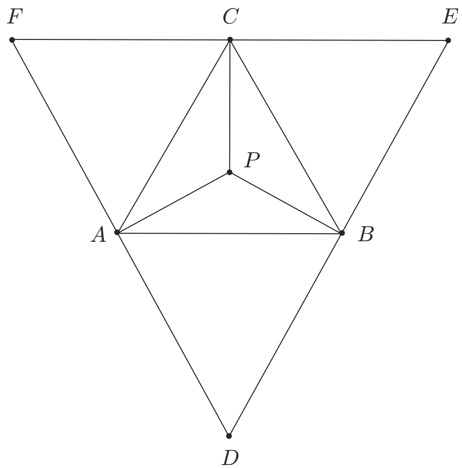

Figure 1.16

Denote by h the length of the altitude of this triangle. Then we know that

$$PA + PB + PC = h.$$

Let M be an arbitrary point on the boundary or in the interior of triangle ABC. Then the sum of distances from M to the sides of triangle DEF is equal